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THEORY OF DETERMINATION OF ULTRA-RADIO FREQUENCIES BY STANDING WAVES ON WIRES.

By August Hund.

ABSTRACT.

This paper furnishes the theoretical background for the method of frequency standardization employing short standing waves on parallel wires. The three essential differences between the parallel-wire system employed by the Bureau of Standards for frequency standardization and the classical system are these: High-frequency voltages are impressed on the wires instead of being caused by discharges originating on the wires; the waves produced are continuous instead of being damped; an electron-tube generating set being substituted for the spark-gap circuit; a current-indicating instrument is used instead of a device for indicating voltage nodes.

The discussion shows that correct results can be obtained by the Bureau of Standard's method if the correction derived in the paper is employed in the calculation of the frequency. It is shown that the adjustment is based on a condition of resonance and, therefore, the settings are sharp, and it is practical to approach the theoretical accuracy in actual operation for frequency standardization.

The main part of the paper gives the practical deductions that may be drawn from the mathematical theory and also the method of calculating the correction to be used in calculating the frequency. The Appendix gives the complete mathematical theory of the action of standing waves on parallel wires with special consideration to the system as used at the Bureau of Standards.

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I. INTRODUCTION.

Accurate knowledge of the frequency of the waves and the currents used in radio communication is of great importance, and the Bureau of Standards has recently increased the accuracy of its standard of radio frequency. The standard has been established by methods[1] which cover a wide range of frequencies and which give the desired accuracy. One[2] of the methods is based on the use of very short waves, the length of which is directly measured on a system of parallel wires. The standard wavemeter may be compared with the frequencies of these short waves by means of the harmonics from a local generating set.

A similar procedure was used in the earliest days of radio.[3] There are, however, three essential differences between the parallel-wire system employed by the Bureau of Standards and the classical two-wire system, namely: (*a*) the high-frequency voltages are impressed on the wires instead of being caused by discharges originating on the wires; (*b*) they are sustained instead of being damped; and (*c*) a current-indicating instrument is used instead of a device for measuring voltage.

It is easier for the experimenter to work with forced sinusoidal waves, since for the open end as well as for the shorted parallel wires a condition exists similar to that met in dealing with standing waves. Any waves traveling on the wires play a part only during the transient state; that is, only during the epoch of

[1] See "Reducing the guesswork in tuning." J. H. Dellinger, Radio Broadcast, 3, p. 241; July, 1923.

[2] "A method of measuring very short radio wave lengths and their use in frequency standardization." F. W. Dunmore and F. H. Engel, Proc. I. R. E. October, 1923, p. 467.

[3] E. Lecher, Wied. Ann. 41, p. 850; 1890. It was also used before by Hertz and Lodge. The Lecher system was also recently used by G. C. Southworth, Radio Review, 1, pp. 577–584; 1920; and 2, pp. 25–31; 1921; J. S. Townsend and J. H. Morrell, Phil. Mag., 42, p. 265; 1921.

switching on the source. It is easier to locate places of current loops or voltage nodes, since ordinary high-frequency measuring instruments can be used and the nodes are decidedly sharp. It is unnecessary to have very high potentials on the parallel wires. The determination is for a condition of resonance; that is, for a state when the forced angular velocity of the induced current is equal to the natural angular velocity of the line. The purpose of this paper is to give the theory of the method and to prove its reliability.

II. THE PARALLEL-WIRE SYSTEM USED BY THE BUREAU OF STANDARDS.[4]

The arrangement is shown in Figure 1. The system of parallel wires is formed by *1*, *2*, *3*, and *4* and coupled to an electron-tube

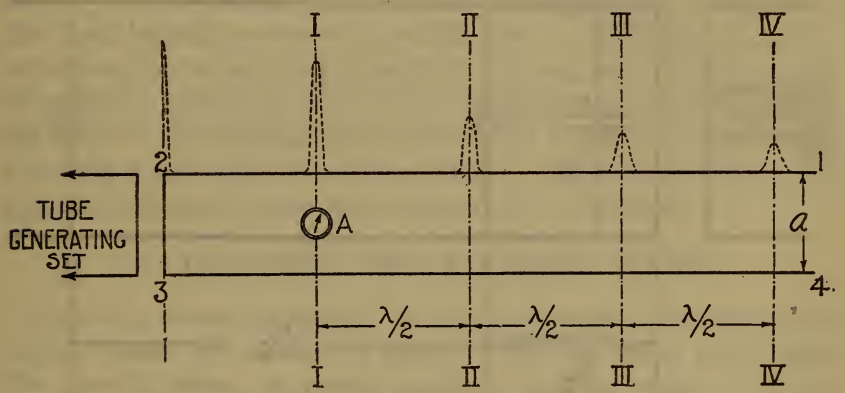


FIG. 1.—*Schematic diagram of parallel-wire system.*

generating set producing currents of ultra-radio frequencies. The parallel wires are open at the end *1*, *4*. A low-resistance current-indicating instrument *A* is connected across the line and can be made to slide along the parallel wires. For a certain position *I*, *I* the indicating instrument *A* gives a pronounced deflection and again when the instrument has been moved to a position *II*, *II*, which is one-half wave length farther from the closed end. The distance *I*, *II* then gives a measure for the frequency of the exciting current. The action on the ammeter is indicated by the dotted curve. The successive maxima decrease somwehat, since a certain amount of energy is used for covering the line losses; that is, to keep the line in a state of oscillation.

[4] For details see "A method of measuring very short radio wave lengths and their use in frequency standardization," by F. W. Dunmore and F. H. Engel, Proc. I. R. E. October, 1923, p. 467.

Measurements show that the distance *II, III* is practically the same as the distance *I, II*. When many maximum conditions are used, there will be small experimental differences in the values for $\lambda/2$. The actual determinations are carried on with the positions *I, I* and *II, II* only, since $\lambda/2$ can be measured most accurately, and the value obtained is reliable. This discussion is to prove the following points:

1. The distance *I, II* is equal to half the wave length corresponding to the frequency of the generating set.

2. It is more accurate to work only with two positions *I, I*, and *II, II*, than with several positions farther out and taking the average of the spacings if there are slight differences for the values of $\lambda/2$.

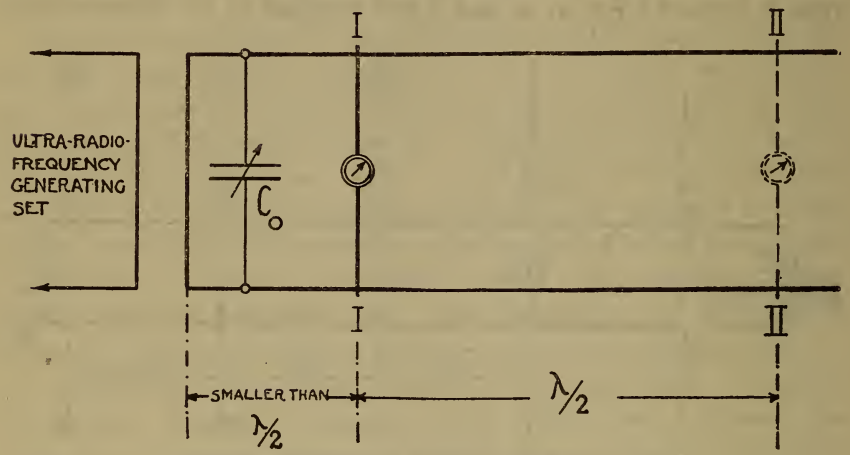


FIG. 2.—*Antinodal points on parallel wires when loaded by means of a condenser.*

3. It is best to keep the line as short as possible and the free end beyond the bridge *II, II* smaller than $\lambda/4$.

4. The spacing *a* between wires has practically no effects on the value of $\lambda/2$ as long as it is not chosen extremely small or extremely large, and the effect of spacing can be calculated.

5. The adjustment is based on a condition of resonance.

6. A condenser C_0 connected across the input terminals (fig. 2) does not affect the distance *I, II*, but shifts the respective resonance settings *I, I* and *II, II* toward the input end.

7. The frequency f may be calculated with high accuracy from the formula

$$f=\frac{v_0}{2l}(1-\Delta)$$

where v_0 is the velocity of light, 2.9982×10^8 meters per second, l is the distance in meters between two consecutive maximum settings and Δ is a small quantity which is a function of the resistance and inductance of the parallel wires at the frequency used.

III. VOLTAGE AND CURRENT FOR ANY CONDITION AT ANY PLACE OF THE PARALLEL WIRES.

Equations (5) and (6) of the Appendix show that the potential and the current at any point of the line are due to traveling waves moving out on the line and returning to the source, respectively. The resulting voltage is therefore equal to the sum of the potential differences of the outgoing and the returning disturbance, while the resulting current is due to the difference of the currents caused by the partial waves. This holds generally; that is, for the open, loaded, or shorted parallel wires. Moreover, the resultant waves for voltage and current of the open and shorted parallel wires have a tendency to turn into standing waves, and the smaller the losses in the line the more pronounced is this tendency. Such a condition can also be produced with an impedance which gives a phase difference of 90° (pure capacity, for instance).

IV. TRAVELING AND STANDING WAVES.

Standing waves represent an electromagnetic hunting, and, strictly taken, no transfer of energy along the wires takes place. The parallel wires are purely reactive, and the voltage and the current are 90° out of phase. Whether the voltage leads or lags, the current depends on the conditions at the receiving end; for example, when the parallel wires are open and a sinusoidal emf is suddenly impressed across the input end, traveling waves will transfer energy along the line (solution 12 of the Appendix) during the transient state, since the voltage and the current are either in phase or differ by 180°. After the energy is supplied to the line a resultant stationary wave will exist along the wires. What happens is that the impressed emf causes a voltage and with it a current impulse to travel toward the open end. They are in phase and carry energy to the different portions of the line. The voltage wave will be reflected at the open end of parallel wires of finite length without change of phase; that is, double the potential at the free end and travel back to the source as though the return were the prolongation of the outgoing wire.

Since no current can exist at the open end, the reflected current wave must necessarily be 180° out of phase and keep on transferring energy. This process goes on until the resulting wave becomes stationary and the current leads the line voltage by 90° (the end of the wires being taken as the origin). It is evident from solution 13 that a similar performance takes place for the wires which are shorted, except that the current wave arriving at the end is reflected without a change in phase and the voltage wave is reflected after a change in phase of 180°. The ultimate state results again in standing waves along the line with the current 90° behind the line voltage. From this it is evident that a condenser across the line will also produce a condition of standing waves as long as its losses are negligible. The distributions along the line are, however, affected by it.

There can be a change of frequency when the traveling waves change over to stationary waves. This is due to the fact that the velocity of propagation of the traveling waves depends only on the line constants; that is, for the parallel-wire system $\left(v=\frac{1}{\sqrt{CL}}\right)$ practically on C and L only.[5] For the stationary state the wires and the coupling transformer to the generating set act as a compound circuit. The slight difference, however, plays no part, since the stationary state is used in the measurement, and a slightly different frequency of the exciting current simply produces a correspondingly different distribution. The wave length measured will, therefore, confirm the frequency of the current of the generating set.

V. FREQUENCY OF THE DIFFERENT TYPES OF PREDOMINATING STANDING WAVES.

For the actual measurements the conditions of shorted parallel wires are utilized. The relation[6] for the effective impedance

$$Z_1 = jZ_0 \tan \beta l \qquad (1)$$

at the input end can therefore be used for finding the frequency of the various modes of predominating standing waves.

$$\beta l = \frac{2\pi}{\lambda} l = \omega l \sqrt{CL} \qquad (2)$$

[5] Strictly speaking, this also includes the direct-current resistance and the frequency, since these quantities are involved in the high-frequency formula for L.

[6] On account of the result obtained in (18a) of the Appendix, the trigonometric formula (15) can be used instead of the expression with tan $h\,nl$.

since

$$\left.\begin{aligned}\lambda &= \frac{v}{f} = \frac{1}{f\sqrt{CL}} \\ \omega &= 2\pi f\end{aligned}\right\} \tag{3}$$

Table 1 gives the results. There are two ways of producing the successive modes of predominating standing waves. One is to keep the frequency $f = f_1$ of the exciting current constant and to choose the length $l = \frac{\lambda}{2}$ or any other integral multiple of it. The

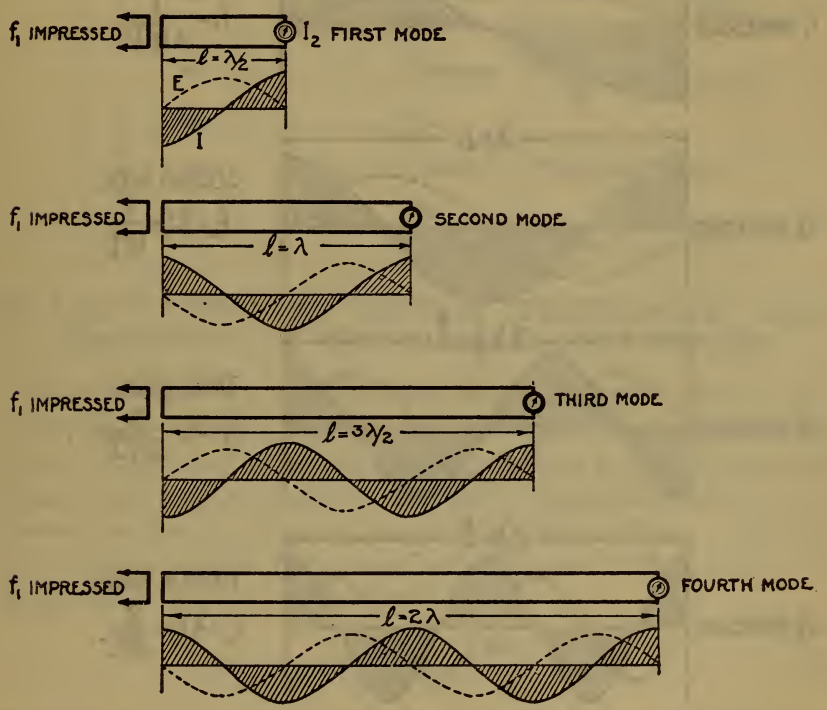


FIG. 3.—*Modes of predominating standing waves with the frequency of the exciting current constant.*

successive modes are then as in Figure 3. For each case a decided maximum current is noted at the shorted end, and a small off-position decreases the reading considerably. The actual length l of the line becomes longer for the higher modes of stationary waves because a larger portion of the wave length due to the impressed frequency f_1 is developed.

The second way is to maintain the physical length l fixed. The frequency $f = f_1$ of the exciting current is then changed in steps,

as is indicated by Figure 4. It is therefore immaterial how the standing waves are produced, since for each case (fig. 3 or fig. 4) the wave length can be measured and used for the calculation of

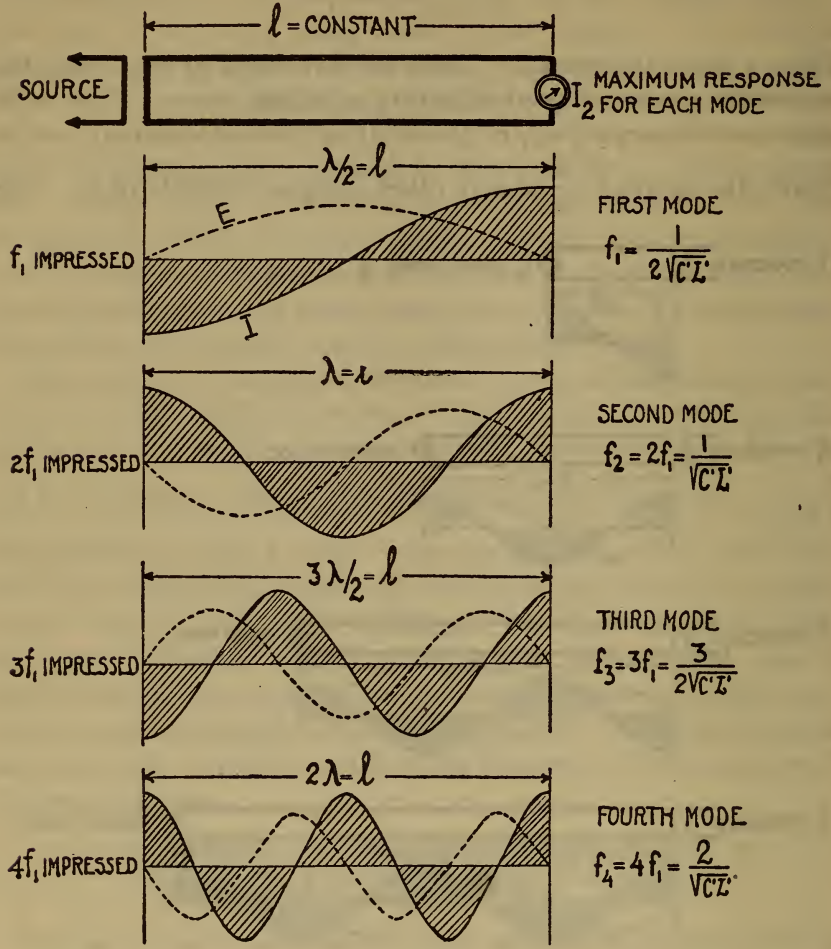


FIG. 4.—*Modes of predominating standing waves with frequency adjusted to* f_1, $2f_1$, $3f_1$, *etc.*

the frequency of the exciting current. From experience the adjustments for Figure 3 seem more suitable for frequency standardization.

TABLE 1.—Frequency and length of parallel-wire system for predominating standing waves.

[C′=lC and L′=lL denote the entire capacity and inductance of parallel wires of length l.]

Actual length l in—	Electrical length.		Effective line impedance Z_l at input end.	Current.	Kind of standing waves.	Formula for the frequency.
	$\omega l\sqrt{CL}$	$\frac{2\pi}{\lambda}l$				
cm	Degrees.	Radians.	Ohms.	Amperes.		Based on $f=\frac{1}{\lambda\sqrt{CL}}$.
$\frac{\lambda}{4}$	90	$\frac{\pi}{2}$	∞	No.		
$\frac{\lambda}{2}$	180	π	0	Maximum.	First mode.	$f_1=\frac{1}{2\sqrt{C'L'}}$
$\frac{3\lambda}{4}$	270	$\frac{3\pi}{2}$	∞	No.		
λ	360	2π	0	Maximum.	Second mode.	$f_2=\frac{1}{\sqrt{C'L'}}=2f_1$
$\frac{5\lambda}{4}$	450	$\frac{5\pi}{2}$	∞	No.		
$\frac{3\lambda}{2}$	540	3π	0	Maximum.	Third mode.	$f_3=3f_1$
$\frac{7\lambda}{4}$	630	$\frac{7\pi}{2}$	∞	No.		
2λ	720	4π	0	Maximum.	Fourth mode.	$f_4=4f_1$

VI. CALCULATION OF NATURAL FREQUENCIES FROM DIMENSIONS OF THE PARALLEL-WIRE SYSTEM.

Experiments show that the wave length measured from the line is practically independent of the spacing and size of the wires. That is equivalent to saying that the various natural frequencies depend only on the length of the line. Expressions will now be derived for the quantitative magnitude of these effects. The relations for the possible frequencies are shown in Table 1 above. It is seen that they are of the form

$$f=\frac{m}{2\sqrt{C'L'}}=\frac{mv}{2l} \tag{4}$$

where $m=1, 2, 3, 4$, etc., for the successive modes of standing waves. Therefore, if the product $C'L'$ is merely a function of the length of the parallel wire, the spacing and the size of the wires can have no effect.

1. ORDINARY FORMULAS FOR C′ AND L′.

Investigating at first the well-known formulas for the mutual[1] capacity C' and the self inductance L' of two parallel bare wires

[1] There is also a capacity action toward the surrounding objects. Experiments show, however, that it has no appreciable effect upon the settings for current maxima. This was proved by first putting the parallel-wire system in the middle of a long room as far as possible from the ceiling, floor, and the walls. Settings taken later on the top of the building over a flat roof were exactly the same. To treat the capacity effects of surrounding objects mathematically would be rather difficult, since no two rooms present the same conditions.

of diameter d_{cm} for a length of l_{cm} in air, and separated a distance of a_{cm} between centers, we have

$$C' = \frac{l}{4 \times 9 \times 10^{11} \, ln \left[\frac{a + \sqrt{a^2 - d^2}}{d} \right]} \text{ farads} = \frac{l}{4 v_0^2 \, ln \frac{1}{b}} \, sec^2/\text{cm} \quad (5)$$

and

$$L' = l \left[4 ln \frac{2a}{d} + 1 \right] 10^{-9} \text{ henries} = \left[4l \cdot ln \frac{2a}{d} + L_i \right] \text{cm} \quad (6)$$

where the expressions to the right are given in electromagnetic cgs units,

$$b = \frac{d/a}{1 + \sqrt{1 - (d/a)^2}}$$

and $v_0 = 2.9982 \times 10^{10}$ cm/sec., and L_i denotes that portion of the inductance which may be considered due to the internal field of the wires.

The capacity formula holds up to the highest frequencies used in radio, and even for the frequencies used in the experiments ($f = 2 \times 10^7$ cycles/sec.). The inductance formula holds for commercial alternating currents; that is, for lower frequencies where the current is uniformly distributed over the cross section of the wire. It is evident that for high-frequency currents of the order of $f = 2 \times 10^7$ cycles/sec. a cylindrical current field must exist to a fair degree of approximation. This suggests the neglect of the inductance L_i due to the flux within the conductors. This would lead to the approximate formula for higher frequencies

$$L' = 4 \times 10^{-9} \, l \cdot ln \frac{2a}{d} \text{ henries} \quad (6a)$$

If the spacing between the wires is not chosen too small, so that the diameter d of the wires is small compared to the spacing a, the capacity formula simplifies to

$$C' = \frac{l}{4 \times 10^{-9} v_0^2 \, ln \frac{2a}{d}} \text{ farads} \quad (5a)$$

With such assumptions

$$\left. \begin{array}{l} C' \cdot L' = \frac{l^2}{v_0^2} \\ \text{or} \\ v_0 = \frac{1}{\sqrt{CL}} \end{array} \right\} \quad (7)$$

and the frequency for any mode of stationary waves would be given by the expression

$$f=\frac{mv_0}{2l}$$

This gives for the half wave length distribution for which $m=1$ and $l=\lambda/2$ the well-known relation

$$f=\frac{v_0}{\lambda}$$

For any other distribution of predominating standing waves—for example, the third mode, $m=3$, and $l=\frac{3\lambda}{2}$—the product of frequency and wave length again confirms the velocity of propagation.

2. HIGH-FREQUENCY FORMULAS FOR C′ AND L′ AND r.

The frequency f must be very much higher than 2×10^7 cycles/sec. before it has any effect upon the distribution of charges over the poles of a condenser. Equation (5) can, therefore, also be considered as the high-frequency formula for C'. The high-frequency inductance is, however, a function of the frequency, of the resistance, and of the dimensions of the parallel wires. It [8] is

$$L'=l\left[4\,ln\frac{1}{b}+\sqrt{\frac{r_0}{\omega\left[1-\left(\frac{d}{a}\right)^2\right]}}\right]cm=l\left[4\,ln\frac{1}{b}+D\right] \qquad (8)$$

all units being in the electromagnetic cgs system and r_0 denoting the direct-current resistance per cm length in cm/sec. of the parallel-wire system. The high-frequency resistance r per cm length of the parallel wires is [8]

$$r=\sqrt{\frac{r_0\omega}{1-\left(\frac{d}{a}\right)^2}} \qquad (8a)$$

If the small quantity D in (8) were neglected, we would again obtain the result expressed in (7), although the assumptions are somewhat different, since nothing needs to be neglected in the

[8] These formulas were derived by Dr. Chester Snow, of the Bureau of Standards, and suggested by him to the writer.

capacity formula. However, taking D into account, we find from (5) and (8)

$$(CL)l^2 = C'L' = \frac{\left[4ln\frac{1}{b} + D\right]l^2}{4v_0^2 ln\frac{1}{b}}$$

Using this in (16b) of the Appendix leads to the true velocity v along the wires

$$v = \frac{1}{\sqrt{CL}} = v_0 \sqrt{\frac{1}{1 + \frac{D}{4\,ln\frac{1}{b}}}} = v_0 \sqrt{\frac{1}{1+2\Delta}} = v_0[1-\Delta]$$

where

$$\Delta = \frac{\sqrt{r_0}}{8\,ln\frac{1}{b}\sqrt{\omega\left[1-\left(\frac{d}{a}\right)^2\right]}} \tag{9}$$

and 2Δ is small compared with unity. The true velocity can therefore be calculated from the formula

$$v = v_0\left[1 - \frac{\sqrt{r_0}}{8\,ln\frac{1}{b}\sqrt{\omega\left[1-\left(\frac{d}{a}\right)^2\right]}}\right] = v_0(1-\Delta) \tag{10}$$

Introducing the high-frequency resistance r from formula (8a) in the expression for Δ gives

$$v = v_0\left[1 - \frac{r}{8\omega ln\frac{1}{b}}\right] \tag{11}$$

or

$$\Delta \sim \frac{r}{2\omega L} \tag{12}$$

VII. ACCURACY OF THE METHOD.

From (12) and (10) it can be seen that the velocity is

$$v = v_0\left[1 - \frac{r}{2\omega L}\right] \tag{13}$$

and from (18a) in the Appendix that the electrical length is

$$\Omega = \pi\left[1 - \left(\frac{r}{2\omega L}\right)^2\right] \tag{14}$$

The velocity is therefore affected by a first order small quantity, while two consecutive maxima settings which are an electrical length Ω apart are affected by a second order small quantity. The error in the method is therefore mostly caused by the deviation of the velocity from the value if there were no attenuation. Formula (10) can, therefore, be used to calculate the correction. Using the actual conditions underlying the experiments, we have: $d = 0.145$ cm; $a = 4.2$ cm; $f = 2 \times 10^7$ cycles/sec., hence $\frac{d}{a} = 0.0345$; $\left(\frac{d}{a}\right)^2 = 0.00119$, which is in this particular case negligible compared with unity. Then, according to (5)

$$\frac{1}{b} = \frac{1 + \sqrt{1 - \left(\frac{d}{a}\right)^2}}{d/a} = 2\,\frac{a}{d} = 57.92$$

Assuming the resistivity of copper equal to 1,600 cgs units, we have, for the two wires of 1 cm length of the parallel-wire system

$$\sqrt{r_0} = \sqrt{\frac{2 \times 1600}{\frac{\pi}{4} 0.145^2}} = 440 \text{ elm. cgs units.}$$

Neglecting, also, the proximity effect in (10) gives the term

$$\Delta = \frac{\sqrt{r_0}}{8\, ln\, \frac{1}{b} \sqrt{2\pi f}} = 0.00121 \text{ and } \Delta^2 = 0.00000146$$

which shows that the effect upon the measured length for $\lambda/2$ can be neglected. The velocity of propagation along the parallel-wire system is therefore only $0.99879\, v_0$ instead of v_0. The values for the frequency which are calculated with the customary velocity $v_0 = 2.9982 \times 10^5$ km/sec. would be 0.121 per cent too high. This error can be avoided by computing the frequency (in kc/sec.) in this particular case from the formula $f = \frac{2.99458 \times 10^5}{\lambda}$ where the wave length is expressed in meters.

It can be seen that the method is reliable when the above correction is taken into account. There seems to be no limitation as to the spacing and the size of the wire as far as the velocity formula (10) is concerned, since for very close spacings the proximity effect is accounted for by the term $\left(\frac{d}{a}\right)^2$ in (10) and the

same term under the square-root value for the rigorous expression of $b=\frac{d/a}{1+\sqrt{1-(d/a)^2}}$. There are, however, practical difficulties. It would not be possible to shift the bridge so that it would always be perpendicular to both wires if the spacing were too wide, and for wide spacings radiation from the wires would increase the attenuation. Too small a spacing would not allow straight connections to the indicating instrument. Moreover, it may be of interest to see to what extent the ordinary inductance formula, if written for the probable high-frequency case (6a), would give a criterion on the accuracy of the method. The error δ in the frequency would then be only due to the assumptions made in the simplified capacity formula (5a) which presupposes the spacing not too close. Any percentage error σ in the total capacity gives $\delta=\frac{\sigma}{2}$, since σ is a small quantity. Applying these results to the actual work for which the spacing $a=4.2$ cm and $d=0.145$ cm, we obtain a frequency which is larger by $\delta=0.00528$ per cent.

This result in comparison with that obtained with the actual high-frequency formula (8) for L' shows that any error due to the capacity is only a second order quantity, and that (6a) can only be considered as an approximate high-frequency formula. The results obtained with (10)—that is, with the correct formula for v—seem to be reliable, since the actual experiments give about the same values[9] as found with other methods.

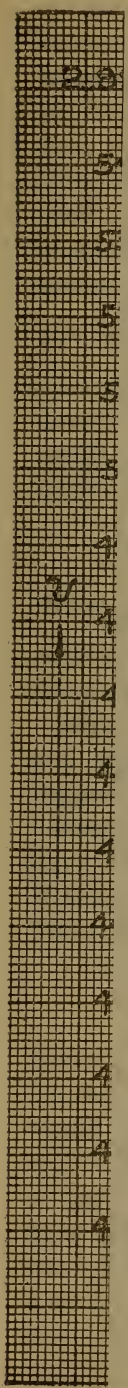
VIII. CORRECTIONS NECESSARY FOR FREQUENCY STANDARDIZATION.

Relative accuracy with respect to other primary standards of one-tenth of a per cent is not satisfactory. It is therefore necessary to use the correction described in the last section. Formula (10) for the reliable velocity v shows that the factor Δ is a function of f. For fixed dimensions of the parallel wires

$$\Delta=\frac{K}{\sqrt{f}} \tag{15}$$

Figure 5 gives the plot for (10) for a spacing of 4.2 cm and copper wires with a diameter of 0.145 cm (No. 15 A. W. G.). It

[9] Calibrations carried on with the parallel-wire system of the above dimensions show that the frequency calculated with v_0 instead of v is about one-tenth of a per cent higher than those obtained with an entirely different method using Lissajou figures and an accurately calibrated tuning fork.



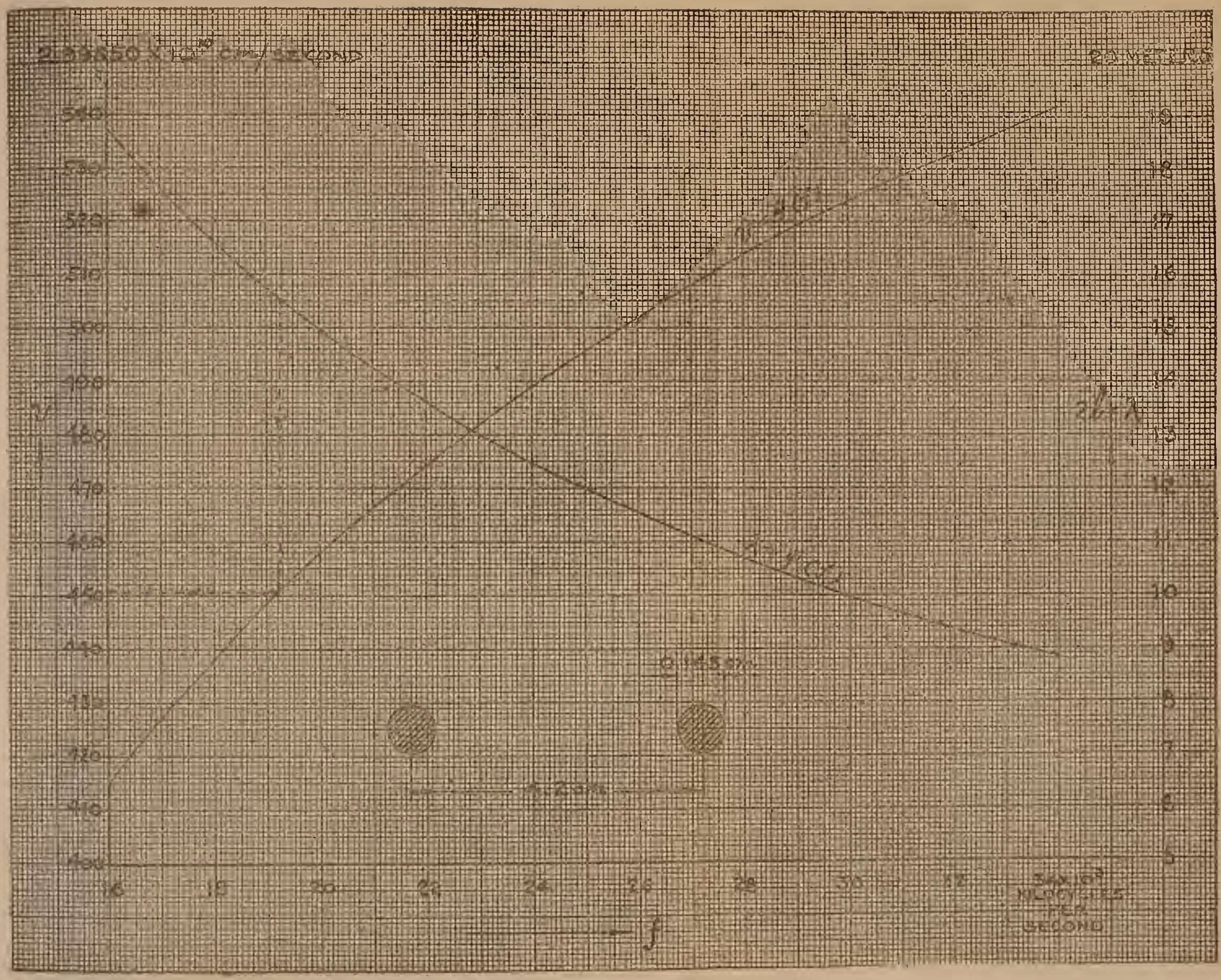


FIG. 5.—*Velocity of propagation for exciting currents of different frequency and a=4.2 cm; d=0.145 cm.*

λ denotes the wave length on the parallel wires and is equal to twice the distance between two consecutive maximum settings.

covers the useful range which is practical for the average indoor laboratory. Since the distance between current antinodes is known at first, it would be necessary to estimate the corresponding frequency by using the value for the velocity in free space and then to look up the velocity for the final calculation of f. To avoid this, the curve $\lambda = \psi(f)$ is drawn into the same figure.[10] It is used as follows: Suppose the distance between two consecutive current maxima is 7,751 mm, then $\lambda = 15.502$ m. This gives, according to the procedure indicated in the figure, $v = 2.994505 \times 10^{10}$ cm/sec. and the standard frequency $f = \frac{2.994505 \times 10^5}{15.502}$ kilocycles/sec. which is also given to a fair degree of approximation directly by the frequency scale. The latter may be used as a rough check. If it is desired to calibrate a standard circuit in terms of the wave length λ_0, it can be found from the relation

$$\lambda_0 = 2l\,(1 + \Delta)$$

where $\Delta = 1 - \frac{v}{v_0}$. This gives, for instance, for above numerical case $2l = 15.502$ m; $v = 2.9945 \times 10^8$ m/sec.; $v_0 = 2.9982 \times 10^8$ m/sec.; $\Delta = 0.00124$; $\lambda_0 = 15.521$ m.

Equation (15) shows that the deviation Δ must become smaller and smaller as f is increased; that means that with increasing frequency the velocity of light is approached. It is therefore of interest to investigate what order of frequency must be used to have a system for which the correction in v can be neglected. If this is done for the particular dimensions used above, we have from (15)

$$\Delta = \frac{5.41}{\sqrt{f}}$$

Hence, even for $\Delta = 0.0001$, the frequency calculated with v_0 instead of v would be 0.01 per cent too high. Such an accuracy would require an exciting current whose frequency is as high as 2.92×10^6 kc/sec. But this corresponds to a wave length of 0.1026 m, which is impractical for this class of work, since for most commercial electron tubes of sufficient power the shortest wave length is in the neighborhood of about 4 m, which requires a series condenser to cut down the capacity of the tube to half of its value.

[10] Strictly speaking, λ is a function of both f and v, since v changes with f.

Similar results would be obtained for other spacings and sizes of wire which are practical for the indicating instrument bridge. The method can, therefore, only be used for frequency standardization when the correction in the velocity is used.

IX. IMPEDANCE CURVES, SPACE RESONANCE CURVES, AND THE EFFECTS ON THE DISTANCE FOR λ/2 FOR MAXIMUM SETTINGS FARTHER OUT.

It is shown in the Appendix (formulas 19, 20, and 20a) that for a constant impressed sinusoidal emf of steady frequency the effective impedance for any electrical length Ω from the indicating instrument to the input end is

$$Z_e = \frac{10^{-9}}{vC}\sqrt{\frac{\Delta^2\Omega^2\cos^2\Omega+\sin^2\Omega}{\Delta^2\Omega^2\sin^2\Omega+\cos^2\Omega}} \quad \text{ohms} \qquad (16)$$

giving for the maximum and minimum values

$$\left.\begin{aligned} Z_{e_{max}} &= \frac{8v_0 ln\,\frac{1}{b}}{(2m-1)\,\pi\Delta}\,10^{-9} \\ Z_{e_{min}} &= \frac{mrv}{4f}\,10^{-9} \end{aligned}\right\} \quad \text{ohms} \qquad (17)$$

if all quantities other than Z_e are expressed in electromagnetic cgs units and m stands for 1, 2, 3, 4, etc., corresponding to successive settings. A glance at these relations shows that the successive points for maximum impedance give $Z_e = \frac{8v_0 ln\,\frac{1}{b}}{\Delta\pi}\,10^{-9} = P$ ohms for $\Omega = \pi/2$ corresponding to the actual length $l = \frac{\lambda}{4}$; $Z_e = \frac{P}{3}$ for $l = \frac{3}{4}\lambda$; $Z_e = \frac{P}{5}$ for $l = \frac{5}{4}\lambda$. The successive maxima are, therefore, in the ratio 1 : 3 : 5 : 7, etc., and of large relative magnitude. For the system used at the Bureau of Standards (No. 15 A. W. G. and a spacing of 4.2 cm) for $f = 2\times10^7$ cycles/sec. the first maximum impedance is $Z_e = 2.56\times10^5$ ohms, the second maximum $\frac{2.56\times10^5}{3}$, the third $\frac{2.56\times10^5}{5}$, etc. Similarly, we see from the above relations that for an electrical length $\Omega = O$ the impedance $Z_e = O$; for $\Omega = \pi$ corresponding to $l = \frac{\lambda}{2}$ the minimum

$Z_e = \frac{rv}{4f} 10^{-9} = Q$ ohms; for $\Omega = 2\pi$ corresponding to $l = \lambda$ an impedance $Z_e = 2Q$; for $l = \frac{3}{2}\lambda$ a value $Z_e = 3Q$ etc. The successive minima are, therefore, twice, three times, four times, etc., as large as the value Q for the half wave length distribution. For the above dimensions and conditions we find $Q = 1.85$ ohms. Hence, the successive minima differ more and more from a zero value, and the settings for maximum current farther out must, therefore, become less sharp.

Experiments show that the settings close to the input end are sharp, and that a small displacement from the resonance setting

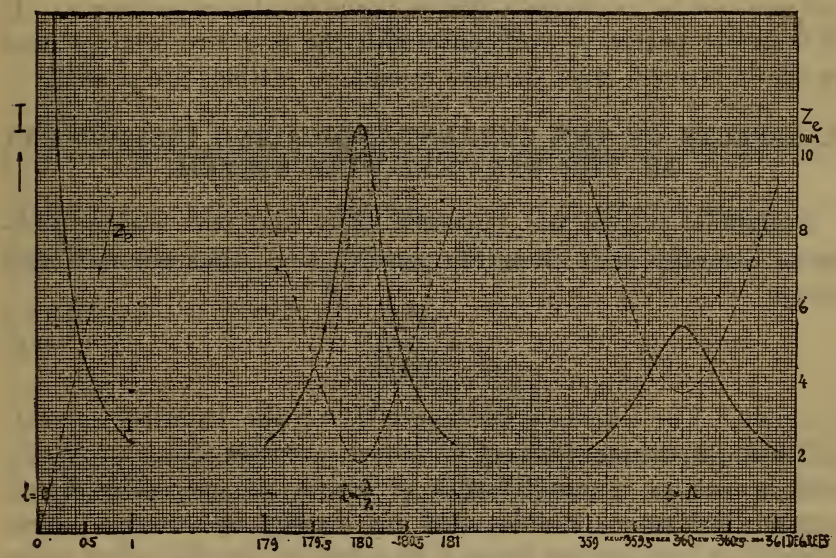


FIG. 6.—*Comparisons of the space-resonance curves for successive modes of line oscillations and a constant impressed frequency.*

$f = 2 \times 10^7$ cycles/sec. for $a = 4.2$ cm and $d = 0.145$ cm.

can be noticed. Therefore, it is only necessary to investigate the course of the impedance curves near the minimum settings. When this is done, we find the results shown in Figure 6. Plotting $\frac{I}{Z_e}$ gives the admittance curves, which, for a constant applied emf of constant frequency f, gives the shape of the space resonance [11] curve [12] for the above dimensions and $f = 2 \times 10^7$

[11] Effective current registered by the ammeter as a function of the displacement from the maximum setting.

[12] The evaluations of (16) show that the resonance curves for $l = \lambda/2$, $l = \lambda$, etc., can not be strictly symmetrical with respect to the ordinate passing through the respective peaks. This dissymetry is, however, so small that it can hardly be noticed for the scales used in the figure. The right-hand part of the resonance curves lies a trifle closer to the ordinate passing through the peak of the curve.

cycles/sec. For a condition of this kind 1° corresponds to a displacement of 41.35 mm, and a shift of the ammeter of 1 mm is noticeable. This is small when compared with the corresponding wave length of $\lambda = 14{,}973$ mm. It is seen that the maximum for the resonance curve for the second mode of line oscillation ($l = \lambda$) is half as high as the maximum noted at $l = \lambda/2$. Actual measurements give a ratio of 0.485, which is a good agreement. The dotted curve is an experimental [13] resonance curve for displacements on each side of $l = \lambda/2$.

It can also be seen that besides the decrease in the successive maximum settings the resonance curves become wider and flatter near the peak. This means that the current antinodes farther out can not be as sharp. This is borne out by experiments. It is therefore better to work only with the first two settings next to the input end, since decided maximum settings are possible.

X. EFFECTS OF THE CURRENT-INDICATING INSTRUMENT ON THE PARALLEL-WIRE SYSTEM.

Equations (14) of the Appendix show that true current antinodes can only exist along the parallel wires when the reflection factor

$$\rho = \frac{R - Z_0}{R + Z_0} \tag{18}$$

is unity. The effective resistance R of the shunted current-indicating instrument is very low, since 0.163 cm (No. 14 A.W.G.) wire is used as a parallel path of length 4.2 cm. The direct-current resistance of this wire and length is 0.000346 ohms. Russell's [14] formula can be used to calculate the skin effect ratio. If $K = \left[\frac{\pi d}{40}\right]^2 \cdot f$ denotes the discriminating constant, it can be written in the abridged form

$$\frac{r_{AC}}{r_{DC}} = 0.5\sqrt{K} + 0.25$$

[13] The experimental resonance curve appears somewhat sharper than the theoretical curve. It is not as easy to take the experimental resonance curve for such a wide range without somewhat affecting the conditions at the input end, which may explain the sharpening effect. The latter is in no way undesirable, since it does not in any way shift the maximum setting which is of interest in the actual work. It is, of course, necessary to use a power tube and a loose coupling.

[14] $\frac{r_{AC}}{r_{DC}} = \frac{\sqrt{K}}{2} + \frac{1}{4} + \frac{3}{32\sqrt{K}} - \frac{1}{16K\sqrt{K}}$ for $K > 5$, which is a modification of Rayleigh's formula and holds only for $K < 1$.

The average frequency for this class of work is $f = 2 \times 10^7$ cycles/sec. Then $\sqrt{K} = 57.2$ and the high-frequency resistance is 28.85 times the steady current resistance; that is, $R = 28.85 \times 0.000346 = 0.00988$ ohms. This is practically the effective resistance across the indicating instrument (4 ohms), since most of the current flows through the shunt instead of through the heater of the indicating instrument.

The surge impedance Z_0, according to deductions given in the Appendix, is

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{r + j\omega L}{j\omega C}}$$

and its absolute amount is calculated from

$$\left| Z_0 \right| = \sqrt{\frac{\sqrt{r^2 + \omega^2 L^2}}{j\omega C}} \qquad (19)$$

Hence, for the dimensions used by the Bureau of Standards $\left(\frac{d}{a}\right)^2$ can be neglected against unity in formulas (5), (8), and (8a), giving $r^2 = 0.0000243$, a negligible quantity compared with $\omega^2 L^2 = 4.18$. This makes $|Z_0| = \sqrt{\frac{L}{C}} = 487.9$ ohms. The reflection factor ρ then becomes, according to (18)

$$\rho = \frac{0.00988 - 487.9}{487.9 + 0.00988}$$

which is practically (-1). The indicating instrument bridging the wire with its shunt can, therefore, be regarded as a short circuit, and the returning wave, reflected at the indicating instrument, starts out with an intensity equal to that of the wave which arrives. This is practically true for the entire range of frequencies which may be standardized with the above system of parallel wires, since the surge impedance is for all useful spacings relatively high compared with the very low effective resistance of the shunted ammeter. This can be seen from[15] Figure 7. Since the bridge can be considered almost as a short circuit, no disturbing

[15] In this figure the surge impedance is calculated from the simplified relation $|Z_0| = 120\, ln\, \frac{2a}{d}$, which is not correct for very small spacings, since the proximity effect $\left(\frac{d}{a}\right)^2$ is neglected in comparison to unity and also the factor D in equation (8). Very small spacings, however, are impractical for this class of work. The neglection of D does not affect the result very much as long as the size of the parallel wires is not chosen unreasonably small. For instance, in the above case $D = 0.03928$ and $4 ln \frac{1}{b} = 16.236$ represents the most significant portion in formula (8). Neglecting D gives 487 ohms instead of 487.9 ohms.

capacity effects due to the indicating instrument can exist when it is set for maximum currents, which conditions are utilized for frequency standardization. That this is true can readily be shown by experiment. A small piece of metal hung on the ammeter or on the wires close by or even as far off as several milli-

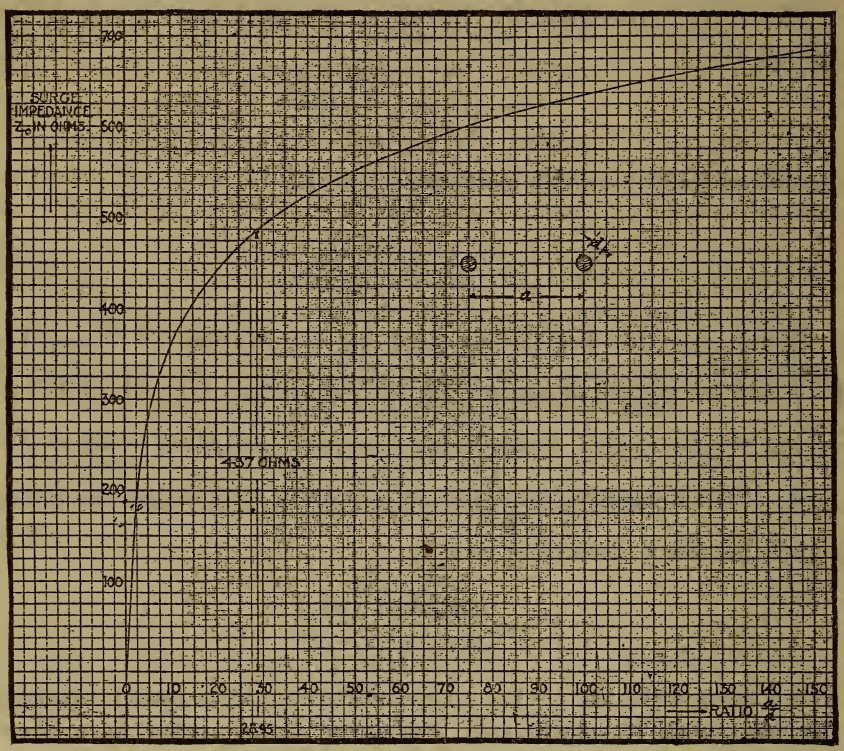


FIG. 7.—*Surge impedance of parallel wires for various values of* $\frac{a}{d}$.

meters along the wires does not affect the setting. It would, of course, disturb the settings if it were placed much farther from the ammeter where higher potentials exist. It has no noticeable effects upon the maximum setting if it is put on the free end behind the ammeter.

XI. EQUIVALENT CIRCUITS WITH LUMPED LINE CONSTANTS, CURRENT, AND VOLTAGE RESONANCE.

Equation (15) of the Appendix shows that the voltage and current distributions of a parallel-wire system when shorted are

$$E = jZ_0 I_2 \sin \beta x$$

$$I = I_2 \cos \beta x$$

This gives an effective impedance Z_1 at the input end

$$Z_1 = jZ_0 \tan \beta l$$

where l is the length of the line in centimeters measured from the shorted end to the source. The voltage and current distributions

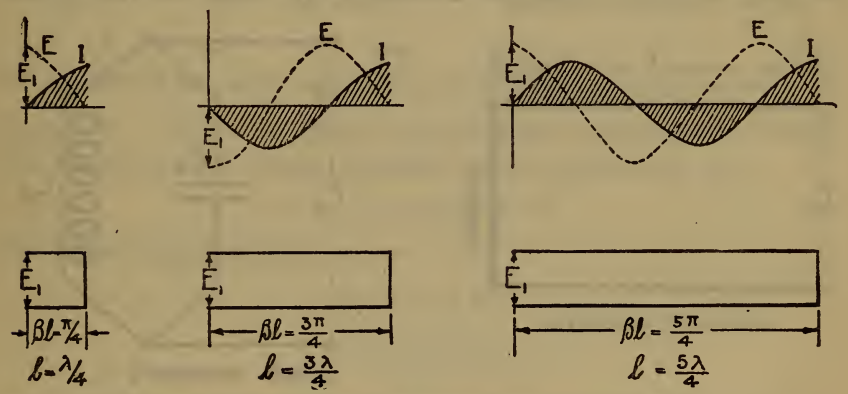


FIG. 8.—*Voltage and current distributions on parallel wires shorted at the end, and* $Z_1 = \infty$.

are drawn in Figure 8, for the first three possible states for which the entrance impedance $Z_1 = \infty$. It is seen that even for large impressed voltages E_1 no current flows into the line. Nevertheless, the line seems to be in a state of standing waves. (This is discussed further under possible standing waves.) For simplicity consider the quarter wave length distribution for which the electrical length $\beta l = \frac{\pi}{4}$. The equivalent circuit with lumped constants of effective values C_e and L_e is shown in Figure 9. It has apparently the same characteristics. A closer inspection shows that no current flows in the main branch for a condition of current resonance. Hence

$$f = \frac{1}{2\pi\sqrt{C_e L_e}}$$

since $\omega L_e = \frac{1}{\omega C_e}$. For the actual system for the quarter wave-length distribution

$$\lambda . f = v = \frac{1}{\sqrt{CL}} \text{ and } l = \frac{\lambda}{4} = \frac{1}{4f\sqrt{CL}}$$

or

$$f = \frac{1}{4\sqrt{C' . L'}}$$

The results for f at first sight do not look alike; but it is to be noted that $C' = l.C$ and $L' = lL$ denote the entire capacity and self-inductance of the closed line of length $l = \frac{\lambda}{4}$. They assume the same potential and the same high-frequency current along the line, while the effective constants are based on the actual

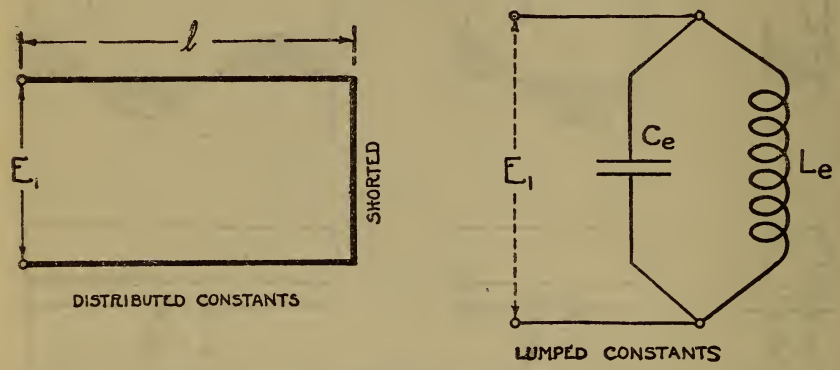


FIG. 9.—*Actual and equivalent circuits for shorted line.*

conditions and, therefore, based on the respective distributions along the wires. The results of solution (22) of the Appendix gives for the above system

$$C_e = \frac{8}{\pi^2} C' \text{ and } L_e = \frac{1}{2} L'.$$

The frequency

$$f = \frac{1}{4\sqrt{C'L'}} = \frac{1}{2\pi\sqrt{C_e L_e}}$$

is, therefore, the same for the actual and the equivalent circuit. The impedance

$$Z_1 = \frac{j\omega L_e}{1 - \omega^2 C_e L_e}$$

of the equivalent circuit must be equal to $Z_1 = jZ_0 \tan \omega l \sqrt{CL}$. From this it is seen that there must exist a decided resonance

peak as the frequency of the exciting current is varied in the neighborhood of the resonance value. In the same way it can be shown that for the parallel wires with open-end conditions can be adjusted which give characteristics similar to ordinary voltage

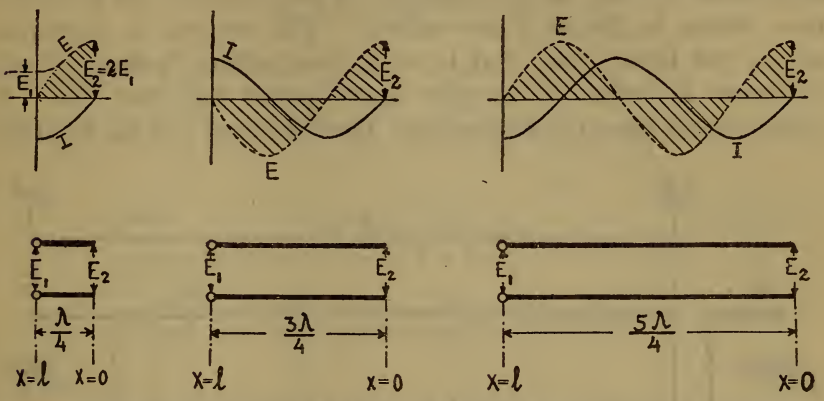


FIG. 10.—*Voltage and current distributions on parallel wires with end open and* $Z^1=O$.

resonance. Figure 10 shows the first three conditions for which the impedance at the end coupled to the generating set,

$$Z_1 = -jZ_0 \operatorname{cotan} \beta l = O$$

The respective distributions are

$$E = E_2 \cos \beta x$$

$$I = j \frac{E_2}{Z_0} \sin \beta x$$

They are easily derived from (7) in the Appendix, since at the open end $I_2 = O$. An infinitesimal voltage from the generating

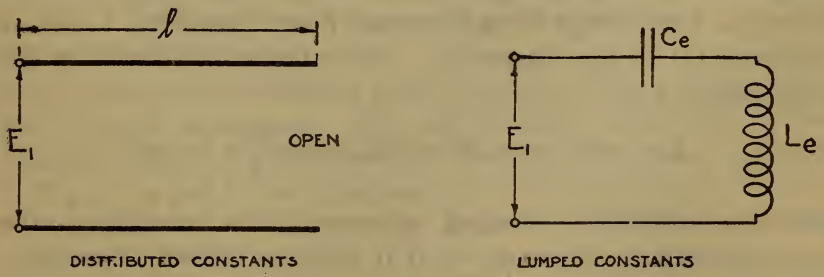


FIG. 11.—*Actual and equivalent circuits for open line standing waves.*

set produces now strong voltage and current waves along the wires. The equivalent circuit is as indicated in Figure 11. The identity

$$Z_0 \operatorname{cotan} \beta l = \frac{1 - \omega^2 C_e L_e}{\omega C_e}$$

must hold, since the impedances for the two circuits are the same.

XII. POSSIBLE STANDING WAVES.

The fact that the length of the line is such that stationary waves are possible does not necessarily mean that these waves are intense enough to be detected. For example, the distributions shown in Figure 8 are correct. The current is, however, zero at the input side; that is, at a place where it should be a maximum in order to transfer much power into the line. A similar case is shown in Figure 12. The frequency f of the exciting

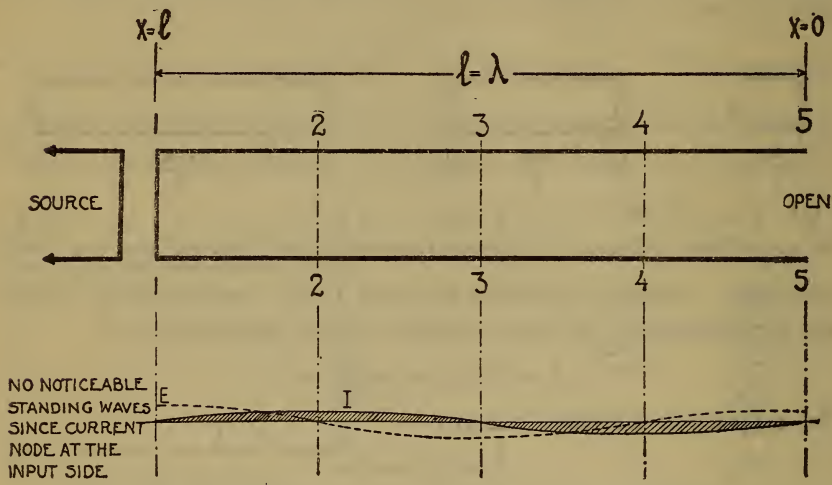


FIG. 12.—*Full-wave length distribution for the open line.*

current is chosen such that the corresponding wave length $\lambda = \frac{1}{f\sqrt{CL}}$ is equal to the entire length l of the wires. The respective distributions for voltage E and current I are, therefore, based on relations (12) of the Appendix. The entrance impedance at the input end *1*, *1* is

$$Z_1 = -jZ_0 \operatorname{cotan} \beta l = -jZ_0 \operatorname{cotan} \frac{2\pi}{\lambda} \cdot \lambda = \infty$$

Hence, no noticeable standing waves can exist, since the entire line acts against the source as if it were an infinite impedance. But when a thermoelectric ammeter of very small resistance (practically a short circuit compared with the surge impedance Z_0) is connected across *5*, *5*, very strong oscillations will be set up. The same thing is true when the ammeter is connected at *3*, *3*, since a decided current reading is noted, while for places somewhat to the right and somewhat to the left almost no current is indi-

cated. The maximum effects at *3, 3* and *5, 5* are so sharp that for $\lambda = 10$ m, for example, a displacement of 1 to 2 mm is noticeable. That resonance takes place at the position *3, 3* can also be understood from

$$Z_1 = jZ_0 \tan \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = O$$

For *5, 5* short circuited

$$Z_1 = jZ_0 \tan \frac{2\pi}{\lambda} \lambda = O$$

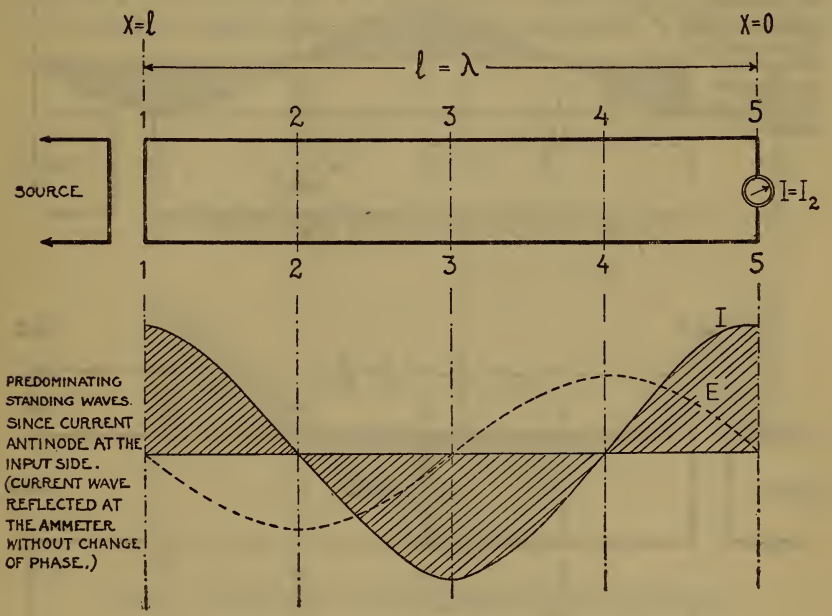


FIG. 13.—*Predominating standing waves on shorted line.*

that is, again zero reactance which accounts for the easily noticeable current maximum. The distributions, therefore, become different when the wires are short circuited. They are indicated in Figure 13 and follow solutions (15) of the Appendix. For the open parallel wires indicated in Figure 14, distributions exist which are given by $E = E_2 \cos \beta x$ and $I = j \frac{E_2}{Z_0} \sin \beta x$. It should, therefore, make no difference to the respective distributions when the wires are shorted by the indicating instrument at places of zero voltage; that is, either at *3, 3* or at *5, 5*. This is, however, not true, because the ammeter of the bridge will read a strong

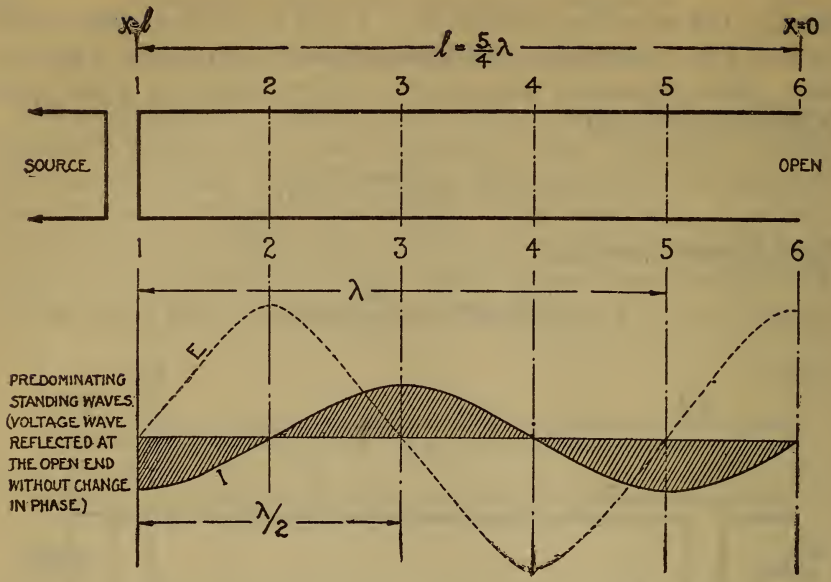


FIG. 14.—*Predominating standing waves on open line.*

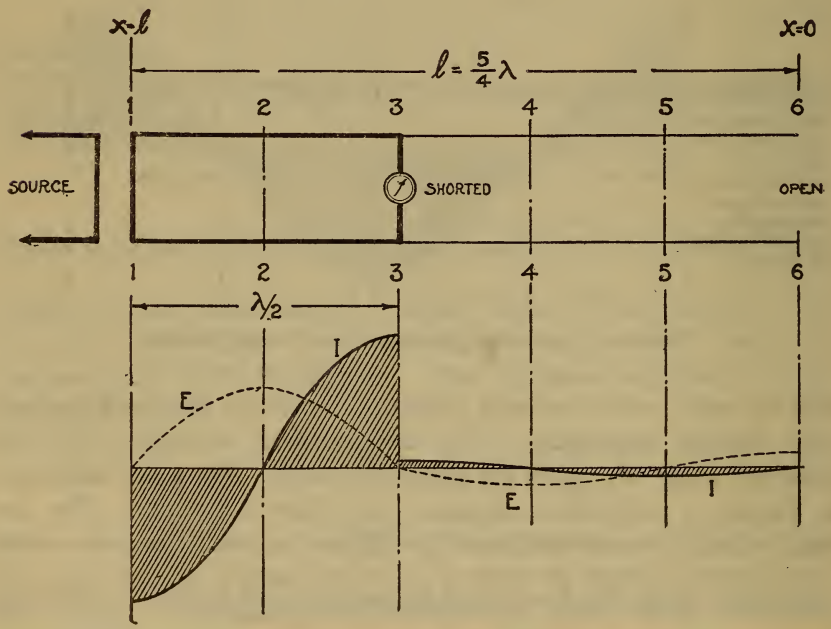


FIG. 15.—*Condition of the standing wave on all parts of the parallel wires, the ammeter being at a voltage node.*

resonance current at either place. Practically the entire energy of the wave is reflected at the ammeter (solution 13) of the Appendix, and for the section *1, 3, 3, 1* there exists, according to equation (15) of the Appendix, an equivalent input impedance

$$Z_1 = jZ_0 \tan \beta \frac{\lambda}{2} = jZ_0 \tan \pi = O$$

The entire section behaves, therefore, like a circuit of zero impedance and allows the flow of a large current. The new distributions are indicated in Figure 15. The distributions behind the ammeter are the same as for the open parallel wires, except that the respective amplitudes are greatly decreased. It is evident

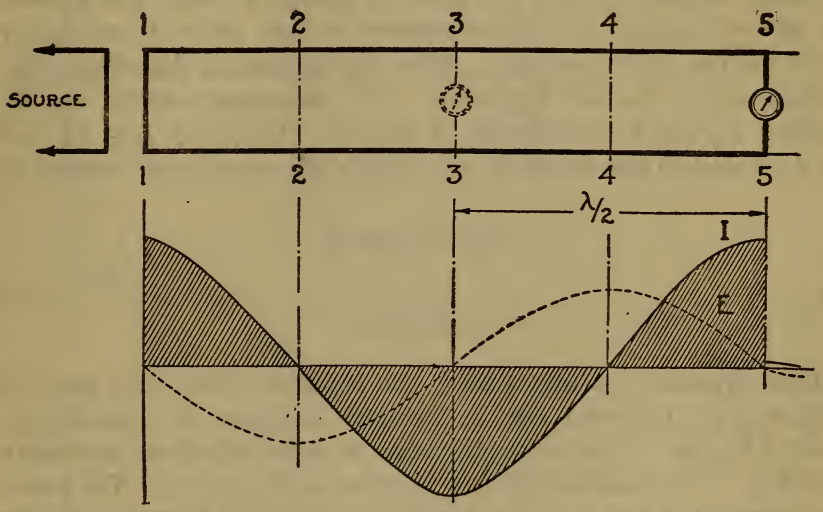


FIG. 16.—*Condition of the standing waves as used for measurement.*

that for a free end equal to $\frac{\lambda}{2}$ the full wave energy is reflected at the ammeter. The actual measurements are, therefore, carried on in such a way that the line was slightly longer than λ. The setting *5, 5* in Figure 16 gives the position $l = \lambda$ with practically no free end effects. The same thing holds true for bridging *3, 3* with the ammeter, since the free end is about $\frac{\lambda}{2}$ away from the ammeter. The distance *3, 5*, therefore, measures half of the wave length of the waves set up by the exciting current. The possible standing waves, therefore, take place when the effective impedance at the input side disappears. For the shorted parallel wires

equation (15) of the Appendix shows that such conditions exist for all values for which $\tan \beta l = O$, hence, for $l = \frac{\lambda}{2}, \lambda, \frac{3}{2}\lambda, 2\lambda$, etc.

XIII. MEANS FOR SHORTENING THE PARALLEL WIRES.

On account of the attenuation and the end effects [16] it is best to use a line as short as possible. This is readily accomplished by the arrangement shown in Figure 2. For certain values of the lumped capacity C_0 the current maximum (I, I) can be brought closer to the input end while the distance I, II still remains half of the wave length. This is proved in the Appendix and is caused by the fact that C_0 is purely reactive. The distributions beyond I, I are the same as though C_0 were not present at all. In many cases the experimenter may prefer to adjust by trial for the best condition. How this is done is described after equation (33) in the Appendix. The shortening of the line by adding C_0 can be calculated, if desired. Equation (30) as well as (32) lead to the result of (33), which is conveniently written as

$$\Omega \frac{C_0}{C'} = \operatorname{cotan} \Omega$$

or

$$f_1(\Omega) = f_2(\Omega) \tag{20}$$

These equations require that C_0 be located at the very end—in this case at the input side, as is shown in Figure 18—and the circuit of Figure 2 is only a modification in order to induce currents in the line. The solution of (20) is given in Figure 17. The intersections of the line $f_1(\Omega)$ for a certain ratio of the lumped capacity C_0 to the distributed capacity $C' = l'C$ give the electrical lengths $\Omega_1, \Omega_2, \Omega_3$, etc. It is only the distance $l' = \frac{\Omega_1}{\beta}$ which is of interest here, since it gives the length to the first current maximum. Since λ can be directly measured as $\frac{\lambda}{2}$ it is convenient to express β in terms of it; that is, by $\beta = \frac{2\pi}{\lambda}$. The distance l' can, therefore, be calculated from the formula

$$l' = \frac{\Omega_1 \lambda}{2\pi} \tag{21}$$

[16] Radiation from an open line.

and the entire minimum length of the parallel wires can be computed from the expression

$$l_{\min} = 0.5\lambda\left[1 + \frac{\Omega_1}{\pi}\right] \tag{22}$$

The numerical case of the transcendental equation (20) is even simpler than shown in Figure 17 because Ω_1 is of interest only and the ascending line $f_1(\Omega)$ intersects the first cotangent curve only on the portion where it is shaded. It is only necessary to draw $f_1(\Omega)$ and $f_2(\Omega)$ for an electrical length of $\frac{\pi}{2}$. The following is an

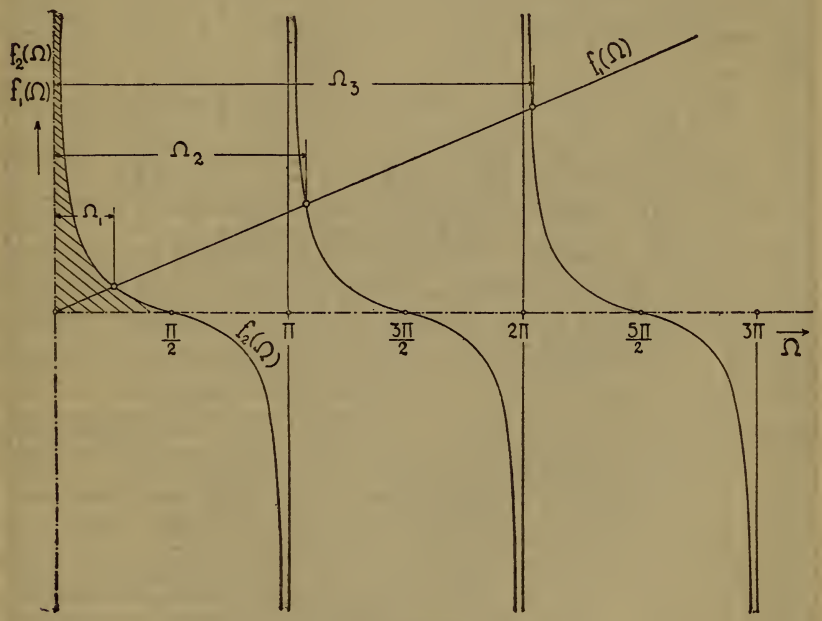


FIG. 17.—*Graphical solution of equation (20).*

illustration: What is the shortening effect for a condenser whose capacity C_0 is equal to the distributed capacity C' corresponding to the length l' to the first voltage node? Draw a cotangent curve $f_2(\Omega)$ from 0 to 90° corresponding to an electrical length $\Omega = \frac{\pi}{2}$ (fig. 18). Since $C_0 = C'$ the absolute value of $f_1(\Omega)$ must be equal to Ω itself. Hence, draw a line $f_1(\Omega)$ such that the ordinate for $\Omega = \frac{\pi}{2}$ is 1.57. The intersection of the two curves gives $\Omega_1 = 0.2778\pi$ and a minimum length for $\lambda = 10$ m of only $l_{\min} = 5(1.2778) = 6.389$ m instead of 10 m. The actual length l' to the

first current response I, I is therefore as short as 1.389 m. This can be confirmed by equation (21). The percentage shortening S can be computed from the formula

$$S\% = 50\left[1 - \frac{\Omega_1}{\pi}\right] \tag{23}$$

For the above example this gives 36.11 per cent. The accuracy of equation (22) can be checked by the limiting conditions. One

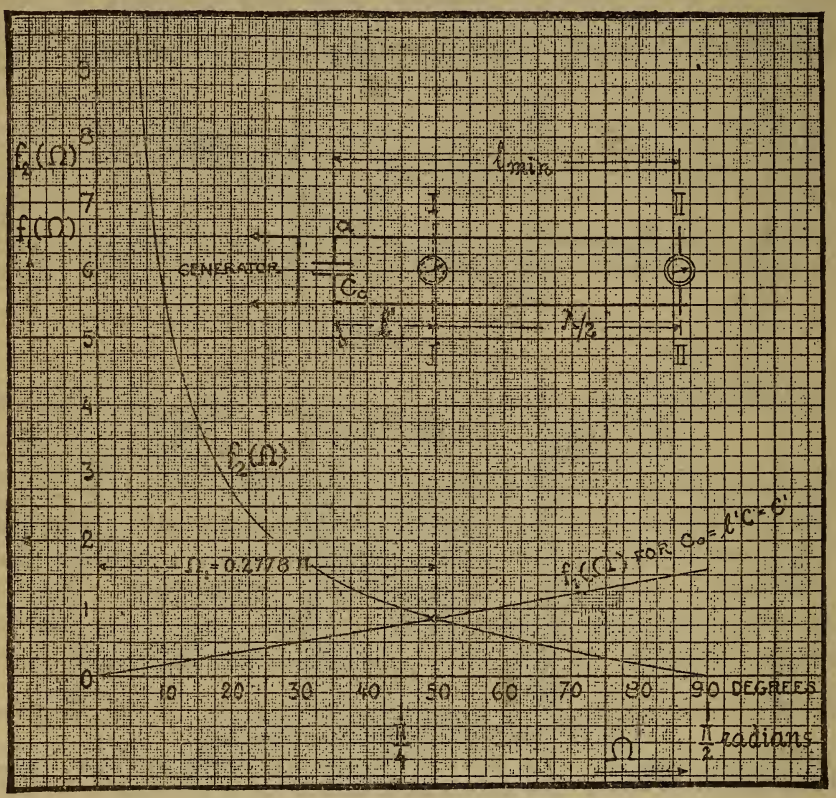


FIG. 18.—*Example of graphical determination of shortening of line produced by addition of C_0.*

limit exists for a very large value of C_0. For a value so large that C_0 becomes a short circuit the ratio $\frac{C_0}{C'}$ becomes infinity, and according to equation (20) we have cotan $\Omega_1 = \infty$ and $\Omega_1 = O$. This can also be seen from Figure 17. As $\frac{C_0}{C_1}$ increases, the ascending line $f_1(\Omega)$ becomes steeper, and for $\frac{C_0}{C'} = \infty$ it coincides with

the vertical coordinate through the origin. The intersection takes place at infinity corresponding to $\Omega = O$. This value inserted in (22) gives $l_{\min} = \frac{\lambda}{2}$ and used in (23) indicates 50 per cent shortening. This is correct, since C_0 is now a short circuit and the line has a voltage node at the beginning of the line and one a distance of $\frac{\lambda}{2}$ farther out where the ammeter bridges the line. To work with such a theoretically ideal condition is not recommended, since any inductive and capacity effects of the coupling to the generating set can have a slight effect on C'. It is better to take another setting farther out and measure the distance *I*, *II* as shown in Figure 1 for obtaining $\frac{\lambda}{2}$. The other limit is for a case where C_0 is infinitely small. Then $\frac{C_0}{C'} = 0$ and the line $f_1(\Omega)$ coincides with the abscissa. The intersection of this axis with $f_2(\Omega)$ gives $\Omega_1 = \frac{\pi}{2}$, and from (22) we obtain $l_{\min} = 0.75\lambda$; that is, a three-fourth wave-length distribution. This is correct again, since an infinitely small condenser C_0 means an open circuit. We have now, according to the diagram of connections in Figure 18, an open line at the input end while the parallel wires are shorted at *II*, *II*. Suppose the system to be excited and C_0 gradually changed to a zero value. Hence, for zero losses the energy would continue to produce standing waves, and for the condition when *a–b* is open the three-fourth wave-length distribution as found above would exist, since for the open end system resonance exists for all values of l which makes cotan βl equal[17] to zero. This does happen for $l = \frac{3}{4}\lambda$, since $\beta = \frac{2\pi}{\lambda}$ and $\beta l = 270°$. This is shown diagrammatically in the second diagram of Figure 8 and shows the case of voltage resonance for the second mode of standing waves.

Since in the actual measurements the distance *I*, *II* shown in Figure 1 is used for finding the value of $\frac{\lambda}{2}$, the portion of *I*, *2*, *3*, *I* can have any shape suitable for coupling to the generating set, and the condenser C_0 can be connected in parallel to *2*, *3* in order to bring the first resonance setting *I*, *I* and all others closer to the input end. The portion *2*, *3* can have a loop shape in a plane

[17] The impedance $Z_1 = -\sqrt{\frac{L}{C}}$ cotan βl disappears for resonance.

perpendicular to the parallel wires. This provides for a coupling which is practically effective only on the portion to which it belongs. The distance *2, I* is then shorter than calculated above, but without any effect on *I, II*.

XIV. CONCLUSIONS.

1. It is shown that the parallel-wire system can be used for primary frequency standardization if a certain small correction is taken into account.

2. The frequency f in kilocycles per second is calculated from a length measured along the wires by the expression

$$f=\frac{v_0}{2l}(1-\Delta)$$

where v_0 is the velocity of light and l denotes the distance in meters between two consecutive maximum settings and Δ is given by equation (9) of the main test. This can be written

$$f=\frac{1.4991\times 10^5}{l}(1-\Delta)$$

The wave length λ_0 of a wave in free space corresponding to this frequency in meters is $2l\ (1+\Delta)$. The quantity $v_0\ (1-\Delta)$ is the velocity of waves along the parallel wires.

3. If many calibrations are to be made and over a wide range of frequencies, a curve may be plotted giving Δ as a function of l. Such a curve would hold for particular dimensions of the parallel-wire system. Another way of facilitating the work is described in Section VII and illustrated in Figure 5.

4. The wave length λ set up on the parallel-wire system is just $2l$, except for a second order small quantity (equation (18a) in the Appendix), but the velocity v with which the wave is propagated in the direction of the parallel wires is not the velocity of light v_0 but a smaller velocity $v=v_0\ (1-\Delta)$. This velocity divided by the true wave length $\lambda=2l$ must be equal to the frequency f.

5. The spacing between the wires and their size have only a slight effect on the observed l, which is taken account of in the correction term. Unless the spacing is chosen extremely small or extremely large this effect is negligible.

6. It is best to keep the line as short as possible and just long enough so that two maximum settings can be made. By connecting a small condenser across the input terminals (fig. 2) both

maximum settings can be moved closer to the input end and without changing the distance between them.

7. The thermoelectric instrument used as detector, which is shunted by a wire of low resistance, does not affect the settings appreciably (Section X).

8. The maximum settings are sharp, since they correspond to a condition of resonance (fig. 6).

In conclusion, the author wishes to acknowledge his indebtedness to Dr. J. H. Dellinger, at whose suggestion the work was undertaken, and for valuable suggestions received from him and Dr. C. Snow in the final preparation of the paper. Acknowledgment is also due to Drs. F. B. Silsbee and C. B. Jolliffe for reading the manuscript and suggestions which have lead to a clearer presentation.

XV. APPENDIX I.

1. DIFFERENTIAL EQUATION AND SOLUTION FOR THE PARALLEL-WIRE SYSTEM.

The problem is the same as confronted by transmission engineers, and, since the general solution using all constants is not much more involved and leads to desirable conclusions, it is given first.

Let r, L, g, and C denote the constants per centimeter length of the parallel wires for the resistance, inductance, leakance, and

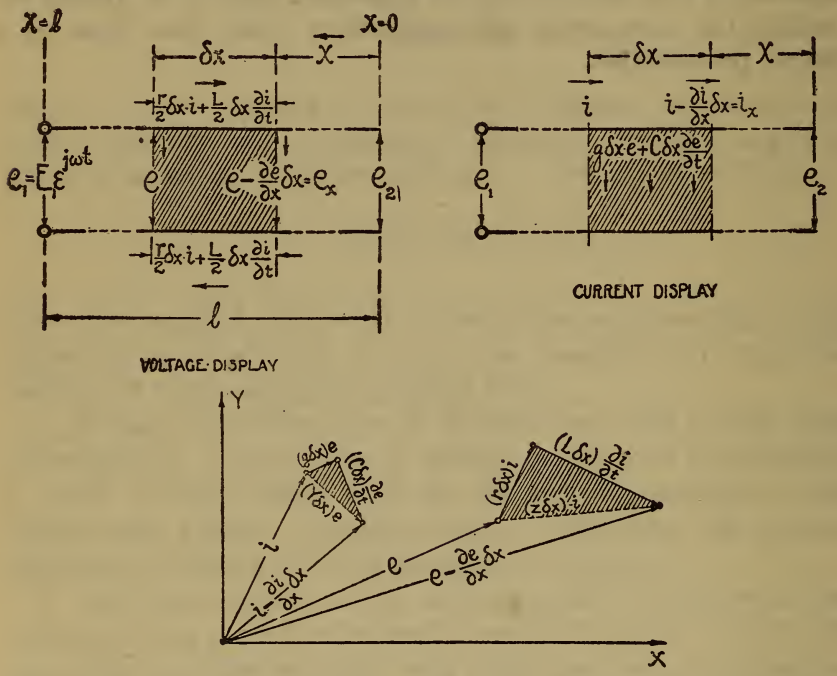


FIG. 19.—*Vector representation of equation (3).*

capacity of the line. Let δx be a small section of the parallel wire as indicated in Figure 19. In order to obtain a simple solution the distance is measured from right toward the source of emf $E_1\epsilon^{j\omega t}$. The voltage drop per unit length $\left[ri+L\frac{\partial i}{\partial t}\right]$ is equal to the decrease $\frac{\partial e}{\partial x}$ of voltage

$$\frac{\partial e}{\partial x}=ri+L\frac{\partial i}{\partial t} \qquad (1)$$

and the loss of current per unit length $\left[ge + C\frac{\partial e}{\partial t}\right]$ must be equal to the decrease $\frac{\partial i}{\partial x}$, hence

$$\frac{\partial i}{\partial x} = ge + C\frac{\partial e}{\partial t} \tag{2}$$

The particular solution of equations (1) and (2), of interest here, is for the case in which both e and i are simple harmonic functions of the time. The instantaneous potential difference e is the real part of $E\epsilon^{j\omega t}$, and the instantaneous value of the current i is the real part of $I\epsilon^{j\omega t}$, where E and I are complex functions of x. The corresponding vector equations of (1) and (2) are [18]

$$\left.\begin{aligned}\frac{\partial E}{\partial x} &= z \cdot I \\ \frac{\partial I}{\partial x} &= y \cdot E\end{aligned}\right\} \tag{3}$$

where $z = r + j\omega L$ is the impedance per unit length of parallel wires and $y = g + j\omega C$ the admittance across the line per unit length. A partial differentiation of equations (3) with respect to x leads to

$$\left.\begin{aligned}\frac{\partial^2 E}{\partial x^2} &= n^2 E \\ \frac{\partial^2 I}{\partial x^2} &= n^2 I\end{aligned}\right\} \quad \ldots\ldots \text{(4) for } n = \sqrt{y \cdot z} = \alpha \pm j\beta$$

The quantity n is known as the propagation constant and the value $\sqrt{\frac{z}{y}} = Z_0$ as the surge impedance of the line. The expression

$$E = a\epsilon^{nx} + b\epsilon^{-nx} \tag{5}$$

is a solution for the first equation of (4), and by means of the first equation of (3) and (5) the solution for the current is

$$I = \frac{1}{Z_0}\left[a\epsilon^{nx} - b\epsilon^{-nx}\right] \tag{6}$$

The quantity n is generally complex, and both relations show that the potential as well as the current at any point of the line

[18] These relations can also be read off directly from the vector diagrams of Figure 19.

is generally caused by wave-like disturbance moving in opposite directions. Suppose the system bridged at the end with an indicating instrument of impedance Z_2. Then there will be at that point for which $x=O$ a voltage $E=E_2$ and a current $I=I_2$ where $e_2=E_2\epsilon^{j\omega t}$ and $i_2=I_2\epsilon^{j\omega t}$. This when used in equations (5) and (6) gives the constants $a=\frac{1}{2}[E_2+Z_0I_2]$ and $b=\frac{1}{2}[E_2-Z_0I_2]$. The voltage and current at any point, therefore, are:

$$\left.\begin{aligned} E&=\frac{1}{2}\{E_2[\epsilon^{nx}+\epsilon^{-nx}]+Z_0I_2[\epsilon^{nx}-\epsilon^{-nx}]\}=E_2\cosh nx+Z_0I_2\sinh nx\\ I&=\frac{1}{2}\{I_2[\epsilon^{nx}+\epsilon^{-nx}]+\frac{E_2}{Z_0}[\epsilon^{nx}-\epsilon^{-nx}]\}=I_2\cosh nx+\frac{E_2}{Z_0}\sinh nx\end{aligned}\right\}(7)$$

For the instrument disconnected $I_2=O$

$$\left.\begin{aligned} E&=E_2\cosh nx\\ I&=\frac{E_2}{Z_0}\sinh nx\end{aligned}\right\}(8)$$

For the line shorted (thermoelectric indicating instrument of very low resistance) $E_2\cong O$

$$\left.\begin{aligned} E&=Z_0I_2\sinh nx\\ I&=I_2\cosh nx\end{aligned}\right\}\text{and } Z=\frac{E}{I}=Z_0\tanh nx \qquad (9)$$

where I_2 denotes the maximum value of the current measured by the ammeter and E and I are the values at any distance x from the instrument.

2. SPECIAL SOLUTION.

When high-frequency currents flow on the parallel wires, the solution becomes simpler. The leakance g across the line can be neglected entirely for two reasons. First, because the two wires are suspended freely in air, and, second, because any leakage effects across the line would be negligible compared with the current component caused by the capacity C of the line. In a similar way r is a small quantity in comparison with ωL. Equations (1) and (2) then give the high-frequency equation

$$\frac{\partial^2 e}{\partial t^2}=v^2\frac{\partial^2 e}{\partial x^2} \qquad (10)$$

A similar expression exists for the current. The solution is

$$\left.\begin{aligned} e &= F(x+vt) + \Phi(x-vt) \\ i &= \frac{F(x+vt) - \Phi(x-vt)}{Z_0} \end{aligned}\right\} \qquad (11)$$

which shows that wave-like disturbances F and Φ move opposite one another with a finite velocity $v = \frac{1}{\sqrt{CL}}$. Its value is somewhat smaller than the velocity in free space. The surge impedance of the double line simplifies with the above assumptions to $Z_0 = \sqrt{\frac{L}{C}}$. The functions F and Φ depend on both time and space and denote traveling waves of unchanged shape. The three cases of importance are:

1. No instrument connected across the end of the line for which the current i vanishes at $x = O$. The second equation of (11) gives

$$F(vt) = \Phi(-vt)$$

Hence

$$\Phi(x-vt) = F(-x+vt)$$

giving for any point x

$$\left.\begin{aligned} e &= F(x+vt) + F(-x+vt) \\ i &= \frac{F(x+vt) - F(-x+vt)}{Z_0} \end{aligned}\right\} \qquad (12)$$

The voltage is therefore higher at any point than when the outgoing disturbance is the only traveling wave.

2. The line is shorted by an instrument of very low impedance (thermoelectric indicating instrument shunted with a heavy copper strip), then the voltage e disappears at $x = O$. The upper equation of (11) gives

$$F(vt) = -\Phi(-vt)$$

Hence

$$\Phi(x-vt) = -F(-x+vt)$$

so that at any point x

$$\left.\begin{aligned} e &= F(x+vt) - F(-x+vt) \\ i &= \frac{F(x+vt) + F(-x+vt)}{Z_0} \end{aligned}\right\} \qquad (13)$$

The current now behaves at any point of the system as the voltage did above.

3. The end of the line is bridged with a thermoelectric ammeter of appreciable resistance R, then at $x=O$ and according to (11)

$$\left.\begin{aligned} e_2 &= F(vt)+\Phi(-vt)=Ri_2 \\ i_2 &= \frac{F(vt)-\Phi(-vt)}{Z_0} \end{aligned}\right\} \qquad (14)$$

or

$$\left.\Phi(-vt)=\frac{R-Z_0}{R+Z_0}F(vt)=\rho F(vt)\right\}$$

so that

$$\Phi(x-vt)=\rho F(-x+vt)$$

Hence, at any point x

$$\left.\begin{aligned} e &= F(x+vt)+\rho F(-x+vt) \\ i &= \frac{F(x+vt)-\rho F(-x+vt)}{Z_0} \end{aligned}\right\}$$

where ρ is the reflection factor caused by the instrument. For $R>Z_0$ the factor ρ is positive and the reflected wave disturbance will have a decreased height, otherwise it behaves as though coming from an open line. For $R<Z_0$ the reflection factor becomes negative and we obtain a reflection similar to the shorted line, but the reflected wave starts out with a smaller amplitude. If the internal resistance of the indicating instrument is equal to the surge impedance Z_0, no reflection takes place, since all the energy of the arriving wave is consumed by the indicating instrument.

3. POTENTIAL AND CURRENT DISTRIBUTIONS WITH THE PARALLEL WIRES SHORTED.

A thermoelectric ammeter with a resistance of 4 ohms was used. Its terminals were shunted by a copper wire in order to reduce the high-frequency resistance to a very small value. Equation (9) can then be used, and for the general case it is seen that the potential and the current distributions toward the input end vary according to hyperbolic sine and cosine functions, respectively; but for the system as used $n=\alpha+j\beta\cong j\beta$ and n can be considered as a circular angle. The electrical length for any distance x is therefore βx. The potential and current distributions

become then circular sine and cosine functions, and equation (9) gives

$$\left.\begin{aligned} E &= jZ_0 I_2 \sin \beta x \\ I &= I_2 \cos \beta x \\ Z &= jZ_0 \tan \beta x \end{aligned}\right\} \qquad (15)$$

the shorted end of the wires being taken as the origin. These relations can be used for calculating the effective line capacity and self-induction and verifying the conditions for current resonance.

4. TIME AND SPACE FUNCTIONS, VELOCITY OF PROPAGATION, AND PROPAGATION CONSTANT.

The relations derived in the preceding section neglected the space dissipation constant α along the line. It was only of interest to obtain expressions for the high-frequency line irrespective of the velocity of propagation. The voltage and current across and through the indicating instrument bridging the wires are in phase, since an ammeter having pure resistance is used. They are, therefore, of the form $E_2 \epsilon^{j\omega t}$ and $I_2 \epsilon^{j\omega t}$. Equation (7) then becomes:

$$\left.\begin{aligned} E &= \frac{\epsilon^{j\omega t}}{2}\{E_2[\epsilon^{nx} + \epsilon^{-nx}] + Z_0 I_2[\epsilon^{nx} - \epsilon^{-nx}]\} \\ I &= \frac{\epsilon^{j\omega t}}{2}\left\{I_2[\epsilon^{nx} + \epsilon^{-nx}] + \frac{E_2}{Z_0}[\epsilon^{nx} - \epsilon^{-nx}]\right\} \end{aligned}\right\} \qquad (7a)$$

Considering the resistance of the instrument very small, then $E_2 = O$ and

$$\left.\begin{aligned} E &= \frac{Z_0 I_2}{2}\{\epsilon^{[\alpha x + j(\omega t + \beta x)]} - \epsilon^{[-\alpha x + j(\omega t - \beta x)]}\} \\ I &= \frac{I_2}{2}\{\epsilon^{[\alpha x + j(\omega t + \beta x)]} + \epsilon^{[-\alpha x + j(\omega t - \beta x)]}\} \end{aligned}\right\} \qquad (7b)$$

(7b) shows that at each point the amplitude varies harmonically with the time, and that similar space distributions exist along the line. The phase of the amplitudes is either ahead or behind by an amount βx with respect to the state at the ammeter. Hence, at points [19] for which x differs by $\frac{2\pi}{\beta} = \lambda$, the same phases exist at the same time. The distance λ corresponds to the time interval

[19] Convenient points of this class are every other current antinode.

of $\frac{1}{f}$ second, if the current energizing the line performs f cycles in the second. The velocity of propagation is then given by

$$v=\frac{\lambda}{1/f}=\frac{\omega}{\beta}=v_0[1-\Delta] \tag{16}$$

This is the velocity along the line and differs by a small amount $\Delta \cdot v_0$ from the velocity $v_0=2.9982 \times 10^5$ km/sec. for electromagnetic waves in empty space. Grouping the real and imaginary terms of equation (4) we have for the propagation constant

$$n=\alpha+j\beta=\sqrt{\tfrac{1}{2}[y \cdot z+r \cdot g-\omega^2 CL]}+j\sqrt{\tfrac{1}{2}[y \cdot z-r \cdot g+\omega^2 CL]} \tag{17}$$

The rigorous formula for the phase velocity then becomes

$$v=\frac{\omega}{\sqrt{\tfrac{1}{2}[y \cdot z-r \cdot g+\omega^2 CL]}} \tag{16a}$$

Bearing in mind that the leakance g across the parallel wires is negligible, we obtain for the propagation constant n the solution

$$n=\sqrt{y \cdot z}=\sqrt{j\omega C[r+j\omega L]}=j\omega\sqrt{CL}\left[1+\frac{r}{j\omega L}\right]\approx j\omega\sqrt{CL}\left[1+\frac{r}{2j\omega L}\right]$$

since $\frac{r}{\omega L}$ is a small quantity and $\left(\frac{r}{\omega L}\right)^2$, etc., can be neglected.

Hence

$$n=\frac{r}{2}\sqrt{\frac{C}{L}}+j\omega\sqrt{CL} \tag{17a}$$

$$=\alpha+j\beta$$

and the velocity may be calculated from the expression

$$v=\frac{\omega}{\beta}=\frac{1}{\sqrt{CL}} \tag{16b}$$

where C and L are the high-frequency capacity and inductance per centimeter length of the parallel wires. This formula gives a means for checking the accuracy of the method if proper formulas for C and L are used.

5. CURRENT THROUGH THE AMMETER BRIDGE FOR CONSTANT IMPRESSED EMF AND FREQUENCY.

Since the experiments are carried on by sliding a shunted thermoelectric indicating instrument of very low resistance along the parallel wires, it is of interest to investigate at what places the current through the indicating instrument becomes a maximum and to what extent the distance between any two consecutive maximum positions is equal to half of the wave length. If l_{cm} stands for the actual length from the bridge to the input end, the electrical length becomes $nl=(\alpha+j\beta)l$. Let E_1 and I_2 denote the maximum values for the impressed emf at the input end and the current through the meter, then the first equation of (7) gives

$$E_1 = Z_0 I_2 \sinh nl$$

If the attenuation constant α were neglected, we would have

$$I_2 = \frac{E_1}{jZ_0 \sin \Omega}$$

and a maximum setting for all values of the simplified electrical length $\Omega=\beta l$, which make sin Ω a minimum. Hence, for $\Omega=\pi$, 2π, 3π, etc., this would confirm that the distance between any two successive maximum settings is equal to $\lambda/2$, since, for example, between the input end and the first maximum setting $\Omega=\pi$ and according to equation (16) $\frac{\lambda \cdot f}{l}=\frac{v}{l}=\frac{\omega}{\beta l}=\frac{2\pi f}{\Omega}$ or $l=\lambda/2$. This is followed up in more detail in Table 1. However, these results can not be accepted without proving that the above assumptions are justified for this particular case. Taking the attenuation into account, we have from the first equation of (7)

$$I_2 = \frac{2yE_1}{n[\epsilon^{nl}-\epsilon^{-nl}]}$$

since $Z_0=\sqrt{\frac{z}{y}}=\frac{\sqrt{yz}}{y}=\frac{n}{y}$. But $y=j\omega C$; $n=\alpha+j\beta$ and $v=\frac{\omega}{\beta}$, hence

$$I_2 = \frac{2vCE_1}{\left[1-j\frac{\alpha}{\beta}\right][\epsilon^{\alpha l}\epsilon^{j\Omega}-\epsilon^{-\alpha l}\epsilon^{-j\Omega}]}$$

Substituting in the ratio $\frac{\alpha}{\beta}$ the values given in (17a), and neglect-

ing insignificant terms in the series for $\epsilon^{\pm\alpha l}$, leads to $\epsilon^{\pm\alpha l} = 1 \pm \alpha l$ and

$$I_2 = \frac{vCE_1}{\left[1 - j\frac{r}{2\omega L}\right][\alpha l \cos \Omega + j \sin \Omega]}$$

Scalarizing this expression, we find

$$|I_2| = \frac{vC|E_1|}{\sqrt{1 + \left(\frac{r}{2\omega L}\right)^2}\sqrt{(\alpha l)^2 \cos^2 \Omega + \sin^2 \Omega}}$$

$$\approx \frac{vC|E_1|}{\sqrt{(\alpha l)^2 \cos^2 \Omega + \sin^2 \Omega}} \qquad (18)$$

This result has the same form if the effective values for $|E_1|$ and I_2 are used. The current through the thermoelectric indicating instrument, therefore, becomes a maximum for all values of Ω which render the function

$$\Psi(\Omega) = (\alpha l)^2 \cos^2 \Omega + \sin^2 \Omega$$

a minimum. This happens for

$$\Omega_m = m\pi\left[1 - \left(\frac{r}{2\omega L}\right)^2\right]$$

$$\approx m\pi \qquad (18a)$$

as is proved below. The distance measured between two consecutive current maxima is therefore equal to π corresponding to $\lambda/2$, since $\left[\frac{r}{2\omega L}\right]^2$ may be neglected in comparison to unity. Proof: Since

$$\alpha l = \frac{r}{2}\sqrt{\frac{C}{L}}\, l = \frac{r\Omega}{2\omega L} = \Delta \cdot \Omega,$$

where Δ is[20] the quantity already used in (16), we have upon differentiating $\Psi(\Omega)$

$$\Psi^1(\Omega) = 2 \cos \Omega\,[\Delta^2\Omega \cos \Omega - \Delta^2\Omega^2 \sin \Omega + \sin \Omega] = O$$

Upon inspecting (18) it is evident that $\cos \Omega = O$ would not produce a maximum in the current, hence the value in the parenthesis must vanish. This leads to the transcendal equation

$$\tan \Omega = -\frac{\Delta^2\Omega}{1 - \Delta^2\Omega^2}$$

[20] This is proved by equations (10) and (11) in the main text.

which can be solved graphically if the dimensions of the parallel wires and the impressed frequency are known. In order to obtain a formula which shows to what extent the value of Ω deviates from π, the method of infinitesimal analysis may be used. The graphical solution indicates that Ω is a very small amount less than π. Hence

$$\Omega_m = m\pi - \kappa_m$$

m being a whole number such as 1, 2, 3, etc. Hence

$$\cos \Omega = \cos (m\pi - \kappa_m) = \cos m\pi \cos \kappa_m = (-1)^m \cos \kappa_m$$

$$\sin \Omega = \sin (m\pi - \kappa_m) = -\cos m\pi \sin \kappa_m = -(-1)^m \sin \kappa_m$$

substituting these results in the expression in the brackets of $\Psi^1(\Omega)$ gives

$$\tan \kappa_m = \frac{\Delta^2 (m\pi - \kappa_m)}{1 - \Delta^2 (m\pi - \kappa_m)^2}$$

expanding results in

$$\kappa_m + \underbrace{\frac{\kappa^3_m}{3} + \frac{2\kappa^5_m}{15}}_{\text{higher terms}} + \cdots = \Delta^2 m\pi + \text{higher terms}$$

or

$$\kappa_m = \Delta^2 m\pi$$

hence

$$\Omega_m = m\pi (1 - \Delta^2)$$

6. IMPEDANCE OF THE PARALLEL-WIRE SYSTEM AND ITS MINIMUM AND MAXIMUM VALUES FOR CONSTANT IMPRESSED EMF AND FREQUENCY.

In (9) we have for the impedance Z of the line experienced at the input end

$$Z_1 = Z_0 \tanh nl = Z_0 \frac{\epsilon^{nl} - \epsilon^{-nl}}{\epsilon^{nl} + \epsilon^{-nl}}$$

But

$$Z_0 = \frac{1 - j \dfrac{r}{2\omega L}}{vC}$$

and neglecting again the insignificant terms in the series for $\epsilon^{\pm \alpha l}$ leads to

$$Z_1 = \frac{1 - j \dfrac{r}{2\omega L}}{vC} \frac{\alpha l \cos \beta l + j \sin \beta l}{\cos \beta l + j \alpha l \sin \beta l}$$

According to the last section $\alpha l = \Delta\Omega$ for $\Omega = \beta l$, and scalarizing the above expression gives

$$\left| Z_1 \right| = \frac{\sqrt{1+\Delta^2}}{vC} \sqrt{\frac{\Delta^2\Omega^2 \cos^2 \Omega + \sin^2 \Omega}{\Delta^2\Omega^2 \sin^2 \Omega + \cos^2 \Omega}}$$

since Δ^2 is a second order small quantity it may be neglected in comparison to unity, and the effective impedance Z_e across the input terminals becomes

$$Z_e = \frac{1}{vC} \sqrt{\frac{\Delta^2\Omega^2 \cos^2 \Omega + \sin^2 \Omega}{\Delta^2\Omega^2 \sin^2 \Omega + \cos^2 \Omega}} \tag{19}$$

Any minimum values of Z_e cause a large current flow into the parallel wires and as such also through the indicating instrument short circuiting the parallel wires. It is immaterial whether the indicating instrument is inserted at the input side; that is, between *2* and *3* of Figure 1 and a straight copper wire of very small resistance is moved out on the parallel wire, or *2* and *3* are connected together as shown in the figure and the bridge is free to move outward. The latter seems more practical for the experimenter, since it is necessary for him to read the indicating instrument while making the maximum settings.

Equation (19) shows that $Z_e \approx \frac{\tan \Omega}{vC}$. Current maxima then occur every half wave length. This is practically true, since $\Delta^2\Omega^2$ is a small [21] quantity. In reality Z_e does not pass through zero values, since both the voltage and current distributions wind themselves around the line coming closest to it every half wave length. The minimum values for Z_e can be found by using the results $\Omega_m = m\pi$ from (18a) in (19) for which

$$Z_{emin} = (-1)^m \frac{mrv}{4f} \text{ cm/sec.} \tag{20}$$

since $\Delta = \frac{r}{2\omega L}$. All quantities are expressed in elm cgs units. The final result is therefore only to be multiplied by 10^{-9} in order to express Z_{emin} in ohms. The maximum values Z_{emax} are found by putting $\Omega_m = (2m-1)\frac{\pi}{2}$.

[21] For the dimensions used in the actual experiments for $f = 2 \times 10^7$ cycles/sec. it is $\Delta^2\Omega^2 = 5.779 \times 10^{-6}$.

Hence

$$Z_{emax} = \frac{2}{(2m-1)\pi\Delta vC}$$

$$\approx \frac{8v_0 ln \frac{I}{b}}{(2m-1)\pi\Delta} \qquad (20a)$$

since

$$\frac{V_0^2}{V} \approx V_0.$$

7. EFFECTIVE LINE CONSTANTS.

According to equation (15) sinusoidal potential and cosinoidal current distributions exist for the shorted parallel wire system. If W_j denotes the power consumed by the heat loss and W_m and W_e the energy of the magnetic and electric field, respectively, the differential equation

$$W_j + \frac{\partial W_e}{\partial t} + \frac{\partial W_m}{\partial t} = O$$

must hold for distributed constants as well as for circuits with lumped constants. It therefore gives a means for finding expressions for the effective line constants since the distributions are known. For the line of length l

$$\left.\begin{aligned} W_j &= \int_0^l rI^2 dx \\ W_e &= \int_0^l \frac{I}{2} CE^2 dx \\ W_m &= \int_0^l \frac{I}{2} LI^2 dx \end{aligned}\right|$$

This gives the expression

$$\frac{\partial I}{\partial t}\int_0^l r \cos^2 \beta x\, dx + \frac{\partial^2 I}{\partial t^2}\int L \cos^2 \beta x\, dx + \frac{I}{\frac{[\int C \sin \beta x\, dx]^2}{\int C \sin^2 \beta x\, dx}} = o$$

which compared with the corresponding differential equation for an ordinary closed circuit

$$\frac{\partial I}{\partial t} r_e + \frac{\partial^2 I}{\partial t^2} L_e + \frac{I}{C_e} = O$$

shows that the effective capacity and inductance of the line can be calculated from the equations

$$\left.\begin{aligned} C_e &= \frac{1}{\beta l}\frac{\left[\int_0^l C' \sin \beta x\, dx\right]^2}{\int_0^l C' \sin^2 \beta x\, dx} \\ L_e &= \frac{1}{\beta l}\int_0^l L' \cos^2 \beta x\, dx \end{aligned}\right\} \text{for } \begin{aligned} C' &= lC \\ L' &= lL \end{aligned} \text{ and any value of } \beta l \tag{21}$$

Applying this to the quarter wave length distribution for which $l=\frac{\lambda}{4}$ and the electrical length of the shorted parallel wires $\beta l=\frac{\pi}{2}$,

$$\left.\begin{aligned} C_e &= \frac{2}{\pi}\frac{\left[\int_0^l C' \sin \beta x dx\right]^2}{\int_0^l C' \sin^2 \beta x\, dx} = \frac{2C'}{\pi}\frac{\left[-\cos \beta x\Big/_0^{x=l}\right]^2}{\frac{\beta x}{2}-\frac{1}{2}\sin \beta x \cos \beta x\Big/_0^{x=l}} = \frac{8}{\pi^2}C' \\ L_e &= \frac{2}{\pi}\int_0^l L' \cos^2 \beta x\, dx = \frac{2L'}{\pi}\left/\frac{\beta x}{2}+\frac{1}{2}\sin \beta x \cos \beta x\right/_0^{x=l} = \frac{L'}{2} \end{aligned}\right\} \tag{22}$$

which means that the effective capacity is only about 81 per cent of the value C′ which holds for a steady potential distribution for the entire line. The effective self-inductance is only half as large as the corresponding line constant L', assuming a uniform high-frequency current along the line. When only quarter and half wave-length distributions or any integer multiples of them are of interest, equation (21) simplifies to

$$\left.\begin{aligned} C_e &= \frac{2C'[1-\cos \beta l]^2}{(\beta l)^2} \\ L_e &= \frac{L'}{2} \end{aligned}\right\} \begin{aligned} &\text{for } \beta l = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi \text{ etc.} \\ &\text{or } l = \frac{\lambda}{4},\ \frac{\lambda}{2},\ \frac{3\lambda}{4},\ \lambda \text{ etc.} \end{aligned} \tag{23}$$

It is interesting to note that for any mode of standing waves either on the open end or on the shorted parallel wires the effective inductance is 50 per cent of the geometrical value since the electrical length βl cancels. It is therefore only the effective capacity C_e which changes in steps from one node to the next. An application is as follows: Suppose the effective capacity of the third representation of Figure 8 is to be found. Then $l=\frac{5\lambda}{4}$; that is,

$\beta l = \frac{5\pi}{2}$ and $C_e = \frac{2C'[1-O]^2}{\left[\frac{5\pi}{2}\right]^2} = \frac{8}{25\pi^2} C'$ and $L_e = \frac{L'}{2}$. The correctness of these results is proved by the following derivation:

$$f = \frac{1}{\lambda\sqrt{CL}} = \frac{1}{\frac{4}{5} l\sqrt{CL}} = \frac{1}{\frac{4}{5}\sqrt{C'L'}} = \frac{1}{\frac{4}{5}\sqrt{\frac{25\pi^2}{8} C_e \cdot 2L_e}} = \frac{1}{2\pi\sqrt{C_e L_e}}$$

Another important result upon comparing differential equations for the lumped and distributed line constants is that the effective resistance of the shorted double line of length l is given by the equation

$$r_e = \frac{1}{\beta l}\int_o^l r' \cos^2 \beta x \, dx \qquad (24)$$

where $r' = rl$. The variable under the integral is the same as for L_e in relation (21). The effective resistance is therefore half as large as the true resistance r', a most desirable result. It shows that the line losses are smaller than anticipated from the value for r', which holds for a uniform high-frequency current along the line

8. EFFECT OF A LUMPED CAPACITY.

Suppose the free end of the parallel wires to be bridged with an air condenser of capacity C_0 which can be considered as purely reactive. If F and Φ now denote the time values of the incident and the reflected disturbances at the condenser, we have from equation (11)

$$E_2 = F + \Phi = \frac{1}{C_0}\int I_2 dt$$

and

$$F - \Phi = I_2 \cdot Z_0$$

This gives

$$2F = I_2 \cdot Z_0 + \frac{1}{C_0}\int I_2 dt \qquad (25)$$

which has the form

$$e = i \cdot r + \frac{1}{C_0}\int i dt \qquad (25a)$$

whose solution is well known. The parallel wires and the condenser act, therefore, as though an emf $2F$ were connected across

a series combination of a resistance $r=Z_0$ and a capacity C_0. First, suppose that F is a rectangular voltage impulse, then

$$I_2=\frac{2F}{Z_0}\epsilon^{-\frac{t}{Z_0C_0}}$$

and

$$\Phi=F-I_2Z_0=F-2F\epsilon^{-\frac{t}{Z_0C_0}} \tag{26}$$

It is of importance to investigate this formula at the instant when Φ just starts out at the condenser C_0 toward the source. This happens at $t=O$ for which

$$\left.\begin{aligned}\Phi&=-F\\ E_2&=O\\ I_2&=\frac{2F}{Z_0}\end{aligned}\right\} \tag{27}$$

At this moment the condenser C_0 acts like a short circuit and produces twice the current which the advancing pulse would do alone. Secondly, suppose a sinusoidal impulse F reaches the condenser C_0. Then

$$I_2=\frac{2F}{Z_1}\left\{\sin\left(\omega t+\tan^{-1}\frac{X_c}{Z_0}\right)+\frac{x_c}{Z_1}\epsilon^{-\frac{t}{Z_0C_0}}\right\}$$

where

$$Z_1=\sqrt{{Z_0}^2-{x_c}^2}\text{ and }x_c=\frac{1}{\omega C_0}$$

At

$$t=O\text{ we have }\Phi=F-\frac{4F\,X_cZ_0}{{Z_1}^2}$$

and

$$\left.\begin{aligned}E_2&=2F\left[1-\frac{2F\cdot X_cZ_0}{{Z_1}^2}\right]\\ I_2&=\frac{4F\cdot x_c}{{Z_1}^2}\end{aligned}\right\}$$

The condenser C_0 does not necessarily act like a short circuit at the first moment, except when $Z_1=\sqrt{x_cZ_0}$. For such an adjustment $I_2=\frac{4F}{Z_0}$ and four times the current passes through C_0 as for F alone. Since C_0 is an air condenser, it consumes a negligible amount of power. The result is that standing waves must be

produced. Equations (7) can be used, and when written for zero attenuation they are

$$\left.\begin{aligned} E &= E_2 \cos \beta x + jZ_0 I_2 \sin \beta x \\ I &= I_2 \cos \beta x + j \frac{E_2}{Z_0} \sin \beta x \end{aligned}\right\}$$

If x denotes the actual length to a point for which there exists a voltage node then

$$I = I_2 \cos \beta x$$

and

$$E_2 \cos \beta x = Z_0 I_2 \sin \beta x = Z_0 E_2 \omega C_0 \sin \beta x$$

or

$$\tan \beta x = \frac{1}{\omega C_0 Z_0} \tag{28}$$

For a small electrical length this becomes

$$x = \frac{1}{\beta \omega C_0 \sqrt{\frac{L}{C}}} = \frac{1}{\omega^2 C_0 L}$$

since $\beta = \omega \sqrt{CL}$

$$\omega = \frac{1}{\sqrt{C_0 (x \cdot L)}} \tag{29}$$

These equations show that voltage resonance exists between the lumped capacity and the distributed inductance along the length x of the double line. The capacity of the line does not, therefore, affect the resonance condition, if it is small compared with C_0. For all other cases equation (28) gives the solution and the length x can be calculated from

$$x = \frac{\tan^{-1} \frac{\sqrt{\frac{C}{L}}}{\omega C_0}}{\omega \sqrt{CL}} \tag{30}$$

It is seen that the parallel wires become shorter when a condenser C_0 is connected across the wires and resonance is adjusted.

When a capacity C_0 is placed next to the input terminals, the distance from this end to the first nodal point of the resultant voltage wire is shortened, and the distribution farther out toward

the shorted end depends only on C and L of the line. The degree of shortening is limited and can be calculated for any value of C_0 when the dimensions of the line are known. The computation is based on the fact that for resonance the net impedance Z_e at the input terminals must vanish. Hence [22]

$$Z_e = jZ_0 \tan \beta x - \frac{j}{\omega C_0} = O \tag{31}$$

Calling l' the actual length from the input end to a setting of the low resistance indicating instrument for maximum response (31) can be brought into the form

$$Z_e = j\sqrt{\frac{L}{C}}\left\{ \tan \beta l' - \frac{C'}{C_0}\frac{1}{\beta l'} \right\} = O \tag{32}$$

since

$$Z_0 = \sqrt{\frac{L}{C}};\ \beta = \omega\sqrt{CL} \text{ and } l'\,C = C'$$

C' denotes the capacity of the double line of length l'. The quantity j belongs to each part and has no effect on the measured values of the quantities involved. It shows only that the entire system of coordinates is rotated backward by 90°. It can, therefore, be ignored. The first term in the parenthesis is a function which repeats itself twice per cycle. Hence, many values of $\beta l'$ will satisfy the equation as is readily seen from the form

$$\frac{C_0}{C'} = \frac{\operatorname{cotan} \beta l'}{\beta l'}$$

$$= \frac{\operatorname{cotan} \Omega_1}{\Omega_1} = \frac{\operatorname{cotan} \Omega_2}{\Omega_2} = \frac{\operatorname{cotan} \Omega_3}{\Omega_3} = \text{etc.} \tag{33}$$

$$\Omega = \beta l$$

It expresses that for a definite ratio of the massed capacity to the distributed capacity resonance can take place for an electrical length Ω_1, Ω_2, Ω_3, etc. The electrical length Ω_1 refers to the first mode, Ω_2 to the second mode, etc., of predominating waves for a case C_0 is connected across the input end. There is an essential difference between these modes and those when no lumped capacity is connected to the line. The ratios $\frac{\Omega_2}{\Omega_1}$, $\frac{\Omega_3}{\Omega_1}$, etc., are no longer straight multiples, but more or less deviating

[22] The first part of equation (31) is given by solution of equation (15).

from the factors 2, 3, etc. A change in Ω can, therefore, be caused by β alone, to l alone, or to both, and they have physical meanings. A change caused by β alone means that a certain line is shorted by an ammeter. A condenser C_0 is connected near the input end and the frequency of the exciting current varied until the first maximum reading is noted. This corresponds to Ω_1. The frequency is gradually increased until another response is noted. This gives Ω_2, and so on. This is a case similar to the one shown in Figure 4, except that in this figure $C_0 = O$ and $\frac{\Omega_2}{\Omega_1} = 2; \frac{\Omega_3}{\Omega_1} = 3$, etc. A change caused by l alone means that the frequency of the exciting current is kept constant, but that the indicating instrument is moved outwards until it reads the first maximum corresponding to Ω_1 then moved on until it gives another response for Ω_2, etc. The case for $C_0 = O$ is shown in Figure 3. The third way of varying Ω by changing both β and l, which means that the frequency of the exciting current is varied while the indicating instrument is moved until the successive maxima are found. This is of little practical value. Another way[23] of changing Ω would be by varying C_0. This is an indirect change, but the most important one for the experimenter. As a matter of fact, it is most convenient to choose a line, say, only 6 to 8 m long for a determination of $\lambda = 10$ m. The procedure is the following: An indicating instrument is connected almost at the end of the parallel wires for an exciting current whose frequency is 2.998×10^4 kilocycles per second. The setting of the capacity C_0 across the input end is varied until a maximum response occurs in the indicating instrument. The position is denoted by *II*, *II*. The indicating instrument is moved gradually toward the input end until at position *I*, *I* it gives another response. From the distance l in meters which the instrument is moved through the frequency $f = \frac{1.499 \times 10^5}{l}$ kilocycles per second can be calculated. From the above discussion it is obvious that the setting *I*, *I* is much closer to the input end than half the wave length, since $\Omega_1 = \beta l'$ is smaller than βl_1, where $\beta l_1 = \beta l$ denotes the first mode of standing waves without C_0 that is caused by C and L of the wires only.

[23] This has reference to the circuit shown in Figure 2.

APPENDIX II.—LIST OF SYMBOLS.

A, a constant.

a, spacing between centers of wires.

$\alpha = \sqrt{\frac{1}{2}[yz + r.g - \omega^2 CL]}$ space attenuation constant.

$b = \frac{\frac{d}{a}}{1 + \sqrt{1 - \left(\frac{d}{a}\right)^2}}$ common term for the capacity and inductance formula. (See (5) and (8).)

$\beta = \sqrt{\frac{1}{2}[yz + rg + \omega^2 CL]}$ wave-length constant of the parallel wires.

$\beta = \omega\sqrt{CL} = \frac{2\pi}{\lambda}$ simplified wave-length constant for the parallel wire system for $g = o$.

βl simplified electrical length for the parallel-wire system.

C, capacity of the parallel wires per centimeter length.

$C' = Cl$ capacity of the parallel wires of length l cm.

C_0 loading capacity at the input end of the system.

C_e effective capacity of the line for actual potential distribution.

$D = \sqrt{\frac{r_0}{\omega[1 - (d/a)^2]}}$ term in the inductance formula which accounts for the attenuation.

d, diameter of the wire.

$\Delta = \frac{\sqrt{r_0}}{8 ln \frac{1}{b}\sqrt{\omega[1 - (d/a)^2]}}$ correction term for the phase velocity and the electrical length.

δ percentage excess of frequency (see page 500).

E maximum value of voltage at any distance x from the end of wires (is a complex function of x).

E_1 maximum voltage at the input.

E_2 maximum voltage at the end of the wires.

e instantaneous value of voltage, the real part $E\,\epsilon^{j\omega t}$.

ϵ base of the natural logarithm (2.71828).

F function symbol.

f frequency.

f_1 frequency of the fundamental current (first mode).

f_2 frequency of the second mode of standing waves $= 2f_1$.

$f_3 = 3f_1$ (third mode), etc.

ϕ function symbol.
g leakance or conductance per cm between the double line.
I maximum value of current at any place x from the end of the wires (a complex function of x).
I_1 current at the input.
I_2 current at the end of the wires.
i instantaneous value of current, the real part of $I\,\epsilon^{j\omega t}$.
$j = \sqrt{-1}$ imaginary unit.
K, k, k_1, etc., constants.
κ very small quantity.
L inductance of the parallel wires per cm of length.
$L' = Ll$ inductance for a length l_{cm}.
L_e effective inductance assuming actual current distribution.
l length of the parallel wires in cm.
$ln = \log_\epsilon$ (natural logarithm).
$\lambda =$ wave length along the wire.
$\lambda_0 =$ wave length in free space.
m whole number, such as 1, 2, 3, etc., otherwise abbreviation for meter.
$n = \alpha + j\beta = \sqrt{yz}$ propagation constant.
$\pi = 3.1416$.
ψ function symbol.
R effective resistance crossing the end of the parallel wires.
r high-frequency resistance of the parallel wires per cm of length.
$r' = rl$ resistance for a length of l_{cm}.
$r_0 =$ direct current resistance per cm length.
$r_e =$ effective resistance assuming actual current distribution.
ρ reflection factor.
S percentage shortening of the line.
σ percentage error in the line capacity.
$\theta =$ ratio $\frac{a}{d}$
$v_0 = 2.9982 \times 10^{10}$ cm/sec. velocity of propagation for zero attenuation.
$v = \frac{1}{\sqrt{CL}}$ velocity of propagation for attenuation, if $g = O$.
Ω electrical length $(= \beta l)$
Ω_1 electrical length for the first mode standing waves.
Ω_2 electrical length for the second mode.
Ω_3 electrical length for the third mode, etc.

$\omega = 2\pi f$ angular velocity of the exciting current.

x distance from the end of the line to any point on the line ($l - x$ from source).

$y = g + j\omega C$ admittance per cm of length of the parallel wires.

$Z = z_0 \tanh nx$ impedance of the line.

$Z_e = \frac{1}{vC}\sqrt{\frac{\Delta^2\Omega^2\cos^2\Omega + \sin^2\Omega}{\Delta^2\Omega^2\sin^2\Omega + \cos^2\Omega}}$ effective impedance of the line across the input terminals.

$z = r + j\omega L$ impedance per cm of length of the parallel wires.

$Z_0 = \sqrt{\frac{z}{y}}$ surge impedance of the line.

WASHINGTON, February, 1924.